

CONVERGENCE OF AN ADAPTIVE MIXED FINITE ELEMENT METHOD FOR GENERAL SECOND ORDER LINEAR ELLIPTIC PROBLEMS

ASHA K. DOND*, NEELA NATARAJ †, AND AMIYA KUMAR PANI ‡

Abstract. The convergence of an adaptive mixed finite element method for general second order linear elliptic problems defined on simply connected bounded polygonal domains is analyzed in this paper. The main difficulties in the analysis are posed by the non-symmetric and indefinite form of the problem along with the lack of the orthogonality property in mixed finite element methods. The important tools in the analysis are *a posteriori* error estimators, quasi-orthogonality property and quasi-discrete reliability established using representation formula for the lowest-order Raviart-Thomas solution in terms of the Crouzeix-Raviart solution of the problem. An adaptive marking in each step for the local refinement is based on the edge residual and volume residual terms of the *a posteriori* estimator. Numerical experiments confirm the theoretical analysis.

Key words. Adaptive mixed finite element method, *a posteriori* error estimator, contraction property, convergence and quasi-optimality.

AMS subject classifications. 65N30, 65N50

1. Introduction. The general second-order linear elliptic PDE on a simply connected bounded polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ reads: For given right-hand side $f \in L^2(\Omega)$, seek u such that

$$(1.1) \quad \mathcal{L}u := -\nabla \cdot (\mathbf{A}\nabla u + u\mathbf{b}) + \gamma u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

The coefficients $\mathbf{A}, \mathbf{b}, \gamma$ are all piecewise smooth and the symmetric matrix \mathbf{A} is positive definite and uniformly bounded away from zero.

The flux variable $\mathbf{p} = -(\mathbf{A}\nabla u + u\mathbf{b})$ and $\mathbf{b}^* = \mathbf{A}^{-1}\mathbf{b}$ allow to recast (1.1) as a first-order system

$$(1.2) \quad \mathbf{A}^{-1}\mathbf{p} + u\mathbf{b}^* + \nabla u = 0 \quad \text{and} \quad \operatorname{div} \mathbf{p} + \gamma u = f \quad \text{in } \Omega.$$

The mixed formulation seeks $(\mathbf{p}, u) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$ such that

$$(1.3) \quad \begin{aligned} (\mathbf{A}^{-1}\mathbf{p} + u\mathbf{b}^*, \mathbf{q}) - (\operatorname{div} \mathbf{q}, u) &= 0 & \text{for all } \mathbf{q} \in H(\operatorname{div}, \Omega), \\ (\operatorname{div} \mathbf{p}, v) + (\gamma u, v) &= (f, v) & \text{for all } v \in L^2(\Omega). \end{aligned}$$

Here and throughout the paper, $H(\operatorname{div}, \Omega) = \{\mathbf{q} \in L^2(\Omega; \mathbb{R}^2) : \operatorname{div} \mathbf{q} \in L^2(\Omega)\}$ and $L^2(\Omega; \mathbb{R}^2)$ denotes the space of \mathbb{R}^2 -valued L^2 functions defined over the domain Ω . The existence and uniqueness of the mixed solution for elliptic problems have been proved in [7, 12].

The study of analysis of the adaptive finite element methods (AFEM) is an essential component of the adaptive process. Various *a posteriori* error estimators are reviewed in [1], and the references therein. The marking strategies, convergence and optimality are well established for the adaptive conforming finite element methods in literature [10, 17, 27, 28, 29, 30, 33]. For the Poisson problem, the convergence and optimality have been established for the adaptive nonconforming FEM [5, 11, 14]

*Department of Mathematics, Indian Institute of Technology, Bombay (asha@math.iitb.ac.in).

†Department of Mathematics, Indian Institute of Technology, Bombay (neela@math.iitb.ac.in)

‡Department of Mathematics, Indian Institute of Technology, Bombay (akp@math.iitb.ac.in)

and for the adaptive mixed FEM [4, 13, 15, 18, 20]. The recent article ‘Axioms of adaptivity’ [16] provides a general framework to optimality of adaptive schemes.

The non-symmetric and indefinite second order elliptic equations with conforming, nonconforming mixed FEM have been discussed in various articles [3, 7, 12, 19, 21, 23, 31, 32]. These articles discuss the existence and uniqueness of the solution with *a priori* error estimates. *A posteriori* error estimates and its convergence for conforming FEM for general second order linear elliptic PDEs have been achieved using contraction of the sum of energy error plus oscillation in [25] and the quasi-optimality in [24]. *A posteriori* error estimates and quasi-optimal convergence of the adaptive nonconforming FEM have been obtained in [21]. To the best of our knowledge, we have not come across any work which discusses the convergence and optimality of the adaptive mixed finite element method (AMFEM) for non-symmetric and indefinite elliptic problems. The main challenges, the lack of orthogonality in MFEM and the non-symmetric form of equation are addressed in this work. Also as the flux variable \mathbf{p} involves u explicitly, the analysis of variable u becomes inevitable for the analysis of the flux \mathbf{p} . In this paper, the main contributions are summarized as:

- for the adaptive algorithm, the marking strategy in each step for the local refinement is proposed based on the comparison of the edge residual term and the volume residual terms of the *a posteriori* estimator,
- *a posteriori* error estimator, quasi-orthogonality property and quasi-discrete reliability results are derived with the help of the representation formula for the lowest-order Raviart-Thomas solution in terms of the Crouzeix-Raviart solution of the problem,
- the contraction property is shown for the linear combination of the sum of errors in \mathbf{p} and u , the edge residual estimator and the volume residual estimators,
- the convergence and the quasi-optimality results are achieved, under the assumption of small initial mesh-size h_0 .

An outline of the paper is as follows. Section 2 introduces notations and the adaptive algorithm for the mixed finite element method. Section 3 describes some auxiliary results necessary for the convergence analysis. The contraction property and the quasi-optimal convergence of the adaptive mixed finite element method are established in Section 4. The numerical experiments are presented in Section 5. Appendix I summarizes the constants used in the article and their interdependencies.

Here are some notations used throughout the paper. An inequality $A \lesssim B$ abbreviates $A \leq CB$, where $C > 0$ is a mesh-size independent constant that depends only on the domain and the shape of finite elements; $A \approx B$ means $A \lesssim B \lesssim A$. Standard notation applies to Lebesgue and Sobolev spaces and $\|\cdot\|$ abbreviates $\|\cdot\|_{L^2(\Omega)}$ with L^2 scalar product (\cdot, \cdot) . For a vector $\mathbf{q} = (q_1, q_2) \in H(\text{div}, \Omega)$, $\|\mathbf{q}\| := (\|q_1\|^2 + \|q_2\|^2)^{1/2}$. Let $\|\cdot\|_T$ and $\|\cdot\|_E$ denote $\|\cdot\|_{L^2(T)}$ and $\|\cdot\|_{L^2(E)}$ respectively. $H^m(\Omega)$ denotes the Sobolev space of order m with norm given by $\|\cdot\|_m$.

2. AMFEM algorithm. This section discusses notations and the adaptive algorithm.

Let \mathcal{T}_h be a regular triangulation the domain $\Omega \subset \mathbb{R}^2$ into triangles such that $\cup_{T \in \mathcal{T}_h} T = \bar{\Omega}$. Let \mathcal{E}_h be the set of all edges in \mathcal{T}_h and let $\mathcal{E}_h(\partial\Omega)$ be the set of all boundary edges in \mathcal{T}_h . Further, let $\text{mid}(E)$ denote the midpoint of the edge E and $\text{mid}(T)$ denote the centroid of the triangle T . The set of edges of the element T is denoted by $\mathcal{E}(T)$ and let $h_T := \text{diam}(T)$ for $T \in \mathcal{T}_h$. Define $h_{\mathcal{T}} \in P_0(\mathcal{T}_h)$, as a piecewise constant mesh-size function such that $h_{\mathcal{T}}|_T := h_T$ for all $T \in \mathcal{T}_h$. Let $h := \max_{T \in \mathcal{T}_h} h_T$ and h_E be the

length of the edge $E \in \mathcal{E}_h$. For any edge E , ν_E is the unit normal vector exterior to T and τ_E is the unit tangential vector along E . Let Π_h be the L^2 projection onto $P_0(\mathcal{T}_h)$ and define $osc_h(f) := \|h_{\mathcal{T}}(1 - \Pi_h)f\|$, where

$$P_r(\mathcal{T}_h) = \{v \in L^2(\Omega) : \forall T \in \mathcal{T}_h, v|_T \in P_r(T)\}.$$

Here, $P_r(T)$ denotes the algebraic polynomials of total degree at most $r \in \mathbb{N}$ as functions on the triangle $T \in \mathcal{T}_h$. The jump of \mathbf{q} across E is denoted by $[\mathbf{q}]_E$; that is, for two neighboring triangles T_+ and T_- ,

$$[\mathbf{q}]_E(x) := (\mathbf{q}|_{T_+}(x) - \mathbf{q}|_{T_-}(x)) \quad \text{for } x \in E = \partial T_+ \cap \partial T_-.$$

The sign of $[\mathbf{q}]_E$ is defined using the convention that there is a fixed orientation of ν_E pointing outside of T_+ . The patch ω_E denote the union of elements that share a common edge E . The piecewise gradient $\nabla_{NC} : H^1(\mathcal{T}_h) \rightarrow L^2(\Omega; \mathbb{R}^2)$ acts as $\nabla_{NC}v|_T = \nabla v|_T$ for all $T \in \mathcal{T}_h$. The broken Sobolev norm $\|\cdot\|_{NC}$ abbreviates $(\mathbf{A} \nabla_{NC} \cdot, \nabla_{NC} \cdot)^{1/2}$.

The non-conforming Crouzeix-Raviart (CR) finite element space with respect to the triangulation \mathcal{T}_h reads

$$\begin{aligned} CR^1(\mathcal{T}_h) &:= \{v \in P_1(\mathcal{T}_h) : v \text{ is continuous in all midpoints } \text{mid}(E) \text{ of edges } E \in \mathcal{E}_h\}, \\ CR_0^1(\mathcal{T}_h) &:= \{v \in CR^1(\mathcal{T}_h) : v(\text{mid}(E)) = 0 \text{ for all } E \in \mathcal{E}_h(\partial\Omega)\}. \end{aligned}$$

The lowest-order Raviart-Thomas space with respect to the triangulation \mathcal{T}_h reads

$$\begin{aligned} RT_0(\mathcal{T}_h) &:= \{\mathbf{q} \in H(\text{div}, \Omega) : \forall T \in \mathcal{T}_h \exists \mathbf{c} \in \mathbb{R}^2 \exists d \in \mathbb{R} \forall \mathbf{x} \in T, \mathbf{q}(\mathbf{x}) = \mathbf{c} + d \mathbf{x} \\ &\quad \text{and } \forall E \in \mathcal{E}_h, [\mathbf{q}]_E \cdot \nu_E = 0\}. \end{aligned}$$

2.1. Algorithm. The standard structure of an adaptive algorithm is successive loops

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}$$

on different levels of the triangulation.

In the step **SOLVE**, the discrete mixed finite element problem (RTFEM) for (1.3) defined by: seek $(\mathbf{p}_h, u_h) \in RT_0(\mathcal{T}_h) \times P_0(\mathcal{T}_h)$ such that

$$(2.1) \quad (\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*, \mathbf{q}_h) - (\text{div } \mathbf{q}_h, u_h) = 0 \quad \text{for all } \mathbf{q}_h \in RT_0(\mathcal{T}_h),$$

$$(2.2) \quad (\text{div } \mathbf{p}_h, v_h) + (\gamma u_h, v_h) = (f_h, v_h) \quad \text{for all } v_h \in P_0(\mathcal{T}_h),$$

is solved. Recall that, $f_h := \Pi_h f$ is the L^2 -projection of f onto $P_0(\mathcal{T}_h)$.

The step **ESTIMATE** consists of computation of an *a posteriori* error estimator. Here, a *a posteriori* error estimator is a combination of the edge estimator η_h and the volume estimator μ_h , that is,

$$(2.3) \quad \|\mathbf{A}^{-1/2}(\mathbf{p} - \mathbf{p}_h)\|^2 + \|u - u_h\|^2 \leq C_{rel}(\eta_h^2 + \mu_h^2), \quad \text{where}$$

$$(2.4) \quad \eta_h^2 := \sum_{E \in \mathcal{E}_h} \eta_h^2(E) \text{ with } \eta_h^2(E) := \|h_E^{1/2}[\mathbf{A}^{-1} \mathbf{p}_h + \mathbf{b}^* u_h] \cdot \tau_E\|_E^2;$$

$$\mu_h^2 := \sum_{T \in \mathcal{T}_h} \mu_h^2(T) \text{ with}$$

$$(2.5) \quad \mu_h^2(T) := osc_h^2(f)_T + \|h \text{div } \mathbf{p}_h\|_T^2 + \|h(\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*)\|_T^2.$$

The *a posteriori* estimate (2.3) is derived in Section 3, Theorem 3.2.

The step MARK consists of the two alternatives **(A)** and **(B)** which depend on the computable quantities η_h and μ_h and a positive parameter κ .

Case (A): if $\mu_h^2 \leq \kappa \eta_h^2$, compute the minimal set of edges $\mathcal{M}_h \subset \mathcal{E}_h$ such that

$$(2.6) \quad \eta_h^2 \leq \eta_h^2(\mathcal{M}_h) \quad \text{with } 0 < \theta_A < 1.$$

Case (B): if $\mu_h^2 > \kappa \eta_h^2$, compute the minimal set of triangles $\mathcal{M}_h \subset \mathcal{T}_h$ such that

$$(2.7) \quad \theta_B \mu_h^2 \leq \mu_h^2(\mathcal{M}_h) \quad \text{with } 0 < \theta_B < 1.$$

Here θ_A , θ_B and κ are the parameters of the marking criteria and will be chosen appropriately.

Newest vertex bisection (NVB) algorithm [6, 33] is applied for refining the marked edges or elements and generate a new regular triangulation in REFINES step. Note that to maintain the conformity of the triangulation, some additional edges and elements may also need refinement.

Remark 2.1. *Instead of separate marking (2.6)-(2.7) defined in MARK, one could use a collective marking, that is, compute the minimal set of triangles $\mathcal{M}_h \subset \mathcal{T}_h$*

$$(2.8) \quad \theta_A(\eta_h^2 + \mu_h^2) \leq (\eta_h^2 + \mu_h^2)(\mathcal{M}_h) \quad \text{with } 0 < \theta < 1.$$

3. Auxiliary results. This section discusses some important results required for the convergence analysis which are the *a posteriori* error estimator, error and estimator reduction properties.

The nonconforming finite element method (NCFEM) for (1.1) seeks $u_{CR} \in CR_0^1(\mathcal{T}_h)$ such that

$$(3.1) \quad (\mathbf{A} \nabla_{NC} u_{CR} + u_{CR} \mathbf{b}, \nabla_{NC} v_{CR}) + (\gamma u_{CR}, v_{CR}) = (f, v_{CR}), \forall v_{CR} \in CR_0^1(\mathcal{T}_h).$$

Representation of RTFEM Solution via NCFEM [12, 26]: The coefficients $\mathbf{A}, \mathbf{b}, \gamma$ are all piecewise constants. Now the auxiliary discrete problem is to seek $u_h^N \in CR_0^1(\mathcal{T}_h)$ such that

$$(3.2) \quad (\mathbf{A} \nabla_{NC} u_h^N + u_h \mathbf{b}, \nabla_{NC} v_{CR}) + (\gamma u_h, v_{CR}) = (f_h, v_{CR}), \text{ for all } v_{CR} \in CR_0^1(\mathcal{T}_h),$$

where for $T \in \mathcal{T}_h$

$$(3.3) \quad u_h(\mathbf{x}) = \left(1 + \frac{S(T)}{4} \gamma\right)^{-1} \left(\Pi_0 u_h^N + \frac{S(T)}{4} f_h\right) \quad \text{for } \mathbf{x} \in T,$$

$$(3.4) \quad S(T) = \int_T (\mathbf{x} - \text{mid}(T)) \cdot \mathbf{A}^{-1}(\mathbf{x} - \text{mid}(T)) d\mathbf{x}.$$

Then, the solution \mathbf{p}_h of the mixed finite element method formulation (2.1)-(2.2) satisfies

$$(3.5) \quad \mathbf{p}_h(\mathbf{x}) = -(\mathbf{A} \nabla_{NC} u_h^N + u_h \mathbf{b}) + (f_h - \gamma u_h) \frac{(\mathbf{x} - \text{mid}(T))}{2} \quad \text{for } \mathbf{x} \in T.$$

The well-posedness of (3.1) and (3.2) and the equivalence of (2.1)-(2.2) with (3.2) is discussed in [12].

LEMMA 3.1. Let u_h^N and (\mathbf{p}_h, u_h) solve (3.2) and (2.1)-(2.2), respectively. Then, it holds

$$(3.6) \quad \|\nabla_{NC} u_h^N\| \leq \|\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*\|,$$

$$(3.7) \quad \|\operatorname{div} \mathbf{p}_h\| = \|f_h - \gamma u_h\| \lesssim \|f_h\| + \|\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*\|.$$

Proof. From (3.5),

$$\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^* = -\nabla_{NC} u_h^N + (f_h - \gamma u_h) \mathbf{A}^{-1} \frac{(\mathbf{x} - \operatorname{mid}(T))}{2}.$$

Since $((f_h - \gamma u_h) (\mathbf{x} - \operatorname{mid}(T)) / 2, \nabla_{NC} u_h^N) = 0$, the Pythagoras theorem yields

$$\|\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*\|^2 = \|\nabla_{NC} u_h^N\|^2 + \|(f_h - \gamma u_h) \mathbf{A}^{-1} \frac{(\mathbf{x} - \operatorname{mid}(T))}{2}\|^2.$$

Hence, (3.6) holds. A use of triangle inequality with (3.3) and (3.6) implies (3.7). \square The following theorem is on *a posteriori* error estimates of $\mathbf{e}_p := \mathbf{p} - \mathbf{p}_h$ and $e_u := u - u_h$ the proof of which is obtained by minor modifications in the proof of Theorem 5.5 in [12]. However, for the sake of completeness, a short proof is given below.

THEOREM 3.2. (*A posteriori error estimate*) Let $u \in H_0^1(\Omega)$ be the unique weak solution of (1.1) and let (\mathbf{p}_h, u_h) be the solution of (2.1)-(2.2). For small initial mesh-size $h_1 > 0$ there holds

$$(3.8) \quad \|\mathbf{A}^{-1/2}(\mathbf{p} - \mathbf{p}_h)\|^2 + \|u - u_h\|^2 \leq C_{rel}(\eta_h^2 + \mu_h^2),$$

where $0 < h \leq h_1$ and η_h, μ_h are as defined in (2.4) and (2.5).

Proof. Consider the Helmholtz decomposition: $\mathbf{e}_p = \mathbf{A} \nabla z + \operatorname{Curl} \beta$ for $z \in H_0^1(\Omega)$ and $\beta \in H^1(\Omega)/\mathbb{R}$. Then

$$(3.9) \quad (\mathbf{A}^{-1} \mathbf{e}_p, \mathbf{e}_p) = (\mathbf{e}_p, \nabla z) + (\mathbf{A}^{-1} \mathbf{e}_p, \operatorname{Curl} \beta).$$

For the first term on the right-hand side of (3.9), an integration by parts with (1.2) and the fact $\operatorname{div} \mathbf{p}_h + \gamma u_h = f_h$ lead to

$$\begin{aligned} (\mathbf{e}_p, \nabla z) &= (\operatorname{div} \mathbf{e}_p, z) = (f - f_h, z) - (\gamma(u - u_h), z) \\ &= (f - f_h, z - \Pi_h z) - (\gamma e_u, z), \\ (3.10) \quad &\lesssim \operatorname{osc}_h(f) \|z\|_1 + \|e_u\| \|z\|. \end{aligned}$$

Define $\beta_h := I_h \beta$, where $I_h : H^1(\Omega) \rightarrow P^1(\mathcal{T}_h) \cap H_0^1(\Omega)$ is the Clement's interpolation operator [34]. With $\operatorname{Curl} \beta_h \in RT_0(\mathcal{T}_h)$, $\operatorname{Curl} \beta_h \perp \nabla H_0^1(\Omega)$ (\perp denotes $L^2(\Omega)$ orthogonality) and (1.2), the second term on the right-hand side of (3.9) can be written as

$$(\mathbf{A}^{-1} \mathbf{e}_p, \operatorname{Curl} \beta) = -(\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*, \operatorname{Curl} (\beta - \beta_h)) - (e_u \mathbf{b}^*, \operatorname{Curl} \beta).$$

From the integration by part formula

$$(3.11) \quad (\mathbf{A}^{-1} \mathbf{e}_p, \operatorname{Curl} \beta) = \sum_{E \in \mathcal{E}_h} \int_E [\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*] \cdot \tau_E (\beta - \beta_h) ds - (e_u \mathbf{b}^*, \operatorname{Curl} \beta).$$

With the interpolation estimates $\|\beta - \beta_h\|_E \leq C h_E^{1/2} \|\beta\|_{1, \omega_E}$, the bounds $\|\nabla \beta\|_{\omega_E} = \|\operatorname{Curl} \beta\|_{\omega_E} \leq \|\mathbf{A}^{-1/2} e_p\|_{\omega_E}$, and $\|z\| \lesssim \|z\|_1 \lesssim \|\mathbf{A}^{-1/2} \mathbf{e}_p\|$, (3.9)-(3.11) result in

$$(3.12) \quad \|\mathbf{A}^{-1/2} \mathbf{e}_p\| \lesssim \operatorname{osc}_h(f) + \|h_E^{1/2} [\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*] \cdot \tau_E\|_{\mathcal{E}_h} + \|e_u\|.$$

To estimate $\|e_u\|$, start with the triangle inequality

$$(3.13) \quad \|e_u\| \leq \|u - u_h^N\| + \|u_h^N - u_h\|.$$

For $\tilde{e} = u_{CR} - u_h^N$, Lemma 4.5 of [12] with sufficiently small mesh-size h shows

$$(3.14) \quad \|\tilde{e}\|_{NC} + \|\tilde{e}\| \lesssim \text{osc}_h(f).$$

For any $\epsilon > 0$, from lemma 3.3 of [12], there exists small mesh-size h such that

$$(3.15) \quad \|u - u_{CR}\| \leq \epsilon \|u - u_{CR}\|_{NC} \quad \text{holds.}$$

A repeated use of the triangle inequality yields estimates for $\|u - u_h^N\|$ in (3.13) as

$$(3.16) \quad \begin{aligned} \|u - u_h^N\| &\leq \|u - u_{CR}\| + \|u_{CR} - u_h^N\| \\ &\lesssim \epsilon (\|u - u_h^N\|_{NC} + \|u_h^N - u_{CR}\|_{NC}) + \|u_{CR} - u_h^N\| \\ &\lesssim \epsilon \|\nabla_{NC}(u - u_h^N)\| + \text{osc}_h(f). \end{aligned}$$

Let $\mathbf{p} = -(\mathbf{A}\nabla u + u\mathbf{b})$ and define $\tilde{\mathbf{p}} = -(\mathbf{A}\nabla_{NC}u_h^N + u_h\mathbf{b})$. Along with an addition and subtraction of the term $\mathbf{A}^{-1}p_h$,

$$(3.17) \quad \|\nabla_{NC}(u - u_h^N)\| \lesssim \|\mathbf{A}^{-1/2}\mathbf{e}_p\| + \|e_u\| + \|\mathbf{p}_h - \tilde{\mathbf{p}}\|.$$

For the third term on the right-hand side of (3.17), (3.5) leads to

$$(3.18) \quad \|\mathbf{p}_h - \tilde{\mathbf{p}}\| \leq \|(f_h - \gamma u_h)(\mathbf{x} - \text{mid}(T))\| \lesssim \|h(f_h - \gamma u_h)\|.$$

The combination of (3.16)-(3.18) results in

$$(3.19) \quad \|u - u_h^N\| \lesssim \text{osc}_h(f) + \epsilon (\|\mathbf{A}^{-1/2}\mathbf{e}_p\| + \|e_u\|) + \epsilon \|h(f_h - \gamma u_h)\|.$$

To bound $\|u_h^N - u_h\|$ in (3.13), use (3.3), the triangle inequality and the fact that $S(T) \approx h^2$ to obtain

$$(3.20) \quad \begin{aligned} \|u_h^N - u_h\| &\leq \left(1 + \frac{S(\mathcal{T})}{4}\gamma\right)^{-1} \|u_h^N - \Pi_0 u_h^N + \frac{S(\mathcal{T})}{4}(\gamma u_h^N - f_h)\| \\ &\lesssim \|h\nabla_{NC}u_h^N\| + \|h^2(u_h^N - u_h)\|_{NC} + \|h^2(f_h - \gamma u_h)\|. \end{aligned}$$

A use of (3.14)-(3.20) in (3.13) along with (2.2) leads to

$$(3.21) \quad \|e_u\| \lesssim \text{osc}_h(f) + \epsilon (\|\mathbf{A}^{-1/2}\mathbf{e}_p\| + \|e_u\|) + \epsilon \|h \operatorname{div} \mathbf{p}_h\| + \|h(\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*)\|.$$

For small mesh-size $h_1 > 0$ with $0 < h \leq h_1$, (3.21) and (3.12) prove (3.8). \square

LEMMA 3.3. (Efficiency) *Let (\mathbf{p}, u) be the solution of (1.3) and (\mathbf{p}_h, u_h) be the solution of (2.1)-(2.2) over the triangulation \mathcal{T}_h . Then, it holds*

$$(3.22) \quad \begin{aligned} C_{\text{eff}}(\|h_E^{1/2}[\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*] \cdot \tau_E\|_{\mathcal{E}_h} + \|h(\mathbf{A}^{-1}\mathbf{p}_h + \mathbf{b}^*u_h)\|) \\ \leq \|u - u_h\| + \|\mathbf{A}^{-1/2}(\mathbf{p} - \mathbf{p}_h)\|. \end{aligned}$$

Proof. The proof is divided into two steps.

Step 1. Let b_E denote the continuous edge bubble function satisfying $0 \leq b_E \leq 1$ on

ω_E and $b_E \in P_2(T)$ for each $T \subset \omega_E$. Let $\phi := [\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*] \cdot \tau_E$ be a polynomial function along E . There exists an extension operator $P : \mathcal{C}(E) \rightarrow \mathcal{C}(\omega_E)$ [34], where $\mathcal{C}(E)$ (resp. $\mathcal{C}(\omega_E)$) denotes the space of the continuous functions defined on E (resp. ω_E) such that the operator P satisfying $P\phi|_E = \phi$ and

$$(3.23) \quad h_E^{1/2} \|\phi\|_E \lesssim \|b_E^{1/2} P\phi\|_{\omega_E} \lesssim h_E^{1/2} \|\phi\|_E.$$

An equivalence of norm argument implies

$$\|\phi\|_E^2 \lesssim \|b_E^{1/2} \phi\|_E^2 = \int_E (b_E P\phi) [\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*] \cdot \tau_E \, ds.$$

An integration by parts and a use of $\text{Curl}(\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*) = 0$ result in

$$(3.24) \quad \|\phi\|_E^2 \lesssim \|b_E^{1/2} \phi\|_E^2 \leq - \int_{\omega_E} \text{Curl}(b_E P\phi) \cdot (\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*) \, ds.$$

Note that $(\nabla u, \text{Curl}(b_E^{1/2} P\phi))_{\omega_E} = 0$. Hence, with the help of the Cauchy-Schwarz inequality, (3.24) can be written as

$$\|\phi\|_E^2 \lesssim \|\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^* + \nabla u\|_{\omega_E} \|\text{Curl}(b_E P\phi)\|_{\omega_E}.$$

The inverse inequality, (3.23) and a utilization of the definition $\mathbf{p} = -(\mathbf{A}\nabla u + u\mathbf{b})$, result in

$$\|h_E^{1/2} [\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*] \cdot \tau_E\|_E \lesssim \|\mathbf{p} - \mathbf{p}_h\|_{\omega_E} + \|u - u_h\|_{\omega_E}.$$

A summation over all the edges leads to an estimate of the first term on the left-hand side of (3.22).

Step 2. Define the function $\mathbf{q}_T := b_T(\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*) \in P_4(T) \cap W_0^{1,\infty}(T)$ and the cubic bubble function $b_T = 27\lambda_1\lambda_2\lambda_3 \in P_3(T) \cap C_0(T)$ in terms of the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$ of $T \in \mathcal{T}_h$ [34]. Since $\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*$ is affine on $T \in \mathcal{T}_h$, an equivalence of norm argument shows

$$\|\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*\|_T^2 \lesssim \int_T \mathbf{q}_T \cdot (\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*) \, dx.$$

The definition of \mathbf{p} and (1.2) show that

$$\|\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*\|_T^2 \lesssim \int_T \mathbf{q}_T \cdot (\mathbf{A}^{-1}(\mathbf{p}_h - \mathbf{p}) - (u - u_h)\mathbf{b}^*) \, dx - \int_T \mathbf{q}_T \cdot \nabla u \, dx.$$

The Cauchy-Schwarz inequality with $\|\mathbf{q}_T\|_T \lesssim \|\mathbf{A}_h^{-1}\mathbf{p}_h + u_h\mathbf{b}_h^*\|_T$ is employed in the first two terms. Adding the zero terms $\nabla u_h|_T$ to the right-hand side of the above equation, an integration by parts shows that

$$\begin{aligned} h_T \|\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*\|_T^2 &\lesssim h_T \|\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*\|_T \left(\|\mathbf{p} - \mathbf{p}_h\|_T + \|u - u_h\|_T \right) \\ &\quad + h_T \int_T (u - u_h) \text{div } \mathbf{q}_T \, dx. \end{aligned}$$

Since $\mathbf{q}_T \in P_4(T)$, an inverse estimate yields

$$h_T \|\text{div } \mathbf{q}_T\|_T \lesssim \|\mathbf{q}_T\|_T \lesssim \|\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*\|_T.$$

Since $h_T \lesssim 1$, it follows

$$h_T \|\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*\|_T \lesssim \|u - u_h\|_T + \|\mathbf{p} - \mathbf{p}_h\|_T.$$

A summation over all elements leads to an estimate of second term on the left-hand side of (3.22). This concludes the proof. \square

Let \mathcal{T}_H and \mathcal{T}_h with $H < h$ denote nested triangulations, and (\mathbf{p}_h, u_h) and (\mathbf{p}_H, u_H) denote the solutions of (2.1)-(2.2) obtained with right-hand sides f_h and f_H over \mathcal{T}_h and \mathcal{T}_H , respectively. The following notations are used in the sequel:

$$\begin{aligned} (3.25a) \quad E_H^2 &:= \|\mathbf{A}^{-1/2}(\mathbf{p}_h - \mathbf{p}_H)\|^2 + \|u_h - u_H\|^2, \\ (3.25b) \quad e_{p_h}^2 &:= \|\mathbf{A}^{-1/2}(\mathbf{p} - \mathbf{p}_h)\|^2, \quad e_{p_H}^2 := \|\mathbf{A}^{-1/2}(\mathbf{p} - \mathbf{p}_H)\|^2, \\ (3.25c) \quad e_{u_h}^2 &:= \|u - u_h\|^2, \quad e_{u_H}^2 := \|u - u_H\|^2, \\ (3.25d) \quad e_h^2 &:= e_{p_h}^2 + e_{u_h}^2, \quad e_H^2 := e_{p_H}^2 + e_{u_H}^2. \end{aligned}$$

LEMMA 3.4. (*Volume estimator reduction*) For given $0 < \theta_B \leq 1$, there exist constants $0 < \delta_1$, $\rho_B < 1$ and the positive constant Λ_2 such that μ_H and E_H defined in (2.5) and (3.25) satisfy

$$(3.26) \quad \mu_h^2 \leq (1 + \delta_1) \mu_H^2 + \Lambda_1 H^2 E_H^2 \quad \text{for the Case (A),}$$

$$(3.27) \quad \mu_h^2 \leq \rho_B \mu_H^2 + \Lambda_1 H^2 E_H^2 \quad \text{for the Case (B).}$$

Proof. For any triangle $K \in \mathcal{T}_H$, which gets refined in the level \mathcal{T}_h , that is, $K \in \mathcal{T}_H \setminus \mathcal{T}_h$, there exist triangles T_1, T_2, \dots, T_J such that $K = T_1 \cup T_2 \dots \cup T_J$. For $K \in \mathcal{T}_H \setminus \mathcal{T}_h$

$$\mu_h^2(K) = \sum_{j=1}^J \left(\|h(f - f_h)\|_{T_j}^2 + \|h \operatorname{div} \mathbf{p}_h\|_{T_j}^2 + \|h(\mathbf{A}^{-1} \mathbf{p}_h + u_h \mathbf{b}^*)\|_{T_j}^2 \right).$$

At least one refinement of T implies $h \leq H/2$, the fact $\|f - f_h\|_{T_j} \leq \|f - f_H\|_{T_j}$ along with the triangle inequality yields

$$\begin{aligned} \mu_h^2(K) &\leq \frac{H^2}{4} \sum_{j=1}^J \left(\|f - f_H\|_{T_j}^2 + \|\operatorname{div}(\mathbf{p}_h - \mathbf{p}_H) + \operatorname{div} \mathbf{p}_H\|_{T_j}^2 \right. \\ &\quad \left. + \|(\mathbf{A}^{-1} \mathbf{p}_H + u_H \mathbf{b}^*) + \mathbf{A}^{-1}(\mathbf{p}_h - \mathbf{p}_H) + (u_h - u_H) \mathbf{b}^*\|_{T_j}^2 \right) \end{aligned}$$

For each $K \in \mathcal{T}_H$, a use of $\|f_h - f_H\|_K \leq \|\Pi_h(1 - \Pi_H)f\|_K \leq \|f - f_H\|_K$ and (2.2), with Young's inequality yields for $\delta_1 > 0$

$$\begin{aligned} \mu_h^2(K) &\leq \frac{H^2}{4} \sum_{j=1}^J \left(\|(f - f_H)\|_{T_j}^2 + 2\|f_h - f_H\|_{T_j}^2 + 3\|\operatorname{div} \mathbf{p}_H\|_{T_j}^2 + 6\|\gamma(u_h - u_H)\|^2 \right. \\ &\quad \left. + (1 + \delta_1) \|\mathbf{A}^{-1} \mathbf{p}_H + u_H \mathbf{b}^*\|_{T_j}^2 + (1 + \frac{1}{\delta_1}) \|\mathbf{A}^{-1}(\mathbf{p}_h - \mathbf{p}_H) + (u_h - u_H) \mathbf{b}^*\|_{T_j}^2 \right) \\ (3.28) \quad &\leq \frac{3}{4} (1 + \delta_1) \mu_H^2(K) + (1 + \frac{1}{\delta_1}) C_1 H^2 E_H^2(K), \end{aligned}$$

where $C_1 = \max\{\|\mathbf{A}^{-1/2}\|_\infty^2, \|\mathbf{b}^*\|_\infty^2 + 6\|\gamma\|_\infty^2\}$. Denote $(1 + \frac{1}{\delta_1})C_1 =: \Lambda_1$.
For $K \in \mathcal{T}_H \cap \mathcal{T}_h$,

$$(3.29) \quad \mu_h^2(K) \leq (1 + \delta_1)\mu_H^2(K) + \Lambda_1 H^2 E_H^2(K).$$

From (3.28) and (3.29), a summation over all triangles implies

$$(3.30) \quad \mu_h^2 \leq (1 + \delta_1)\mu_H^2 + \Lambda_1 H^2 E_H^2,$$

for the **Case (A)**.

For the **Case (B)**, a summation over all triangles, the marking criteria for $\mathcal{M}_H \subset \mathcal{T}_H$,

$$(3.31) \quad \theta_B \mu_H^2 \leq \mu_H^2(\mathcal{M}_H) \leq \mu_H^2(\mathcal{T}_H \setminus T_h).$$

A use of (3.28), (3.29) and (3.31) leads to the sharper bound

$$\begin{aligned} \mu_h^2 &= \mu_h^2(\mathcal{T}_H \setminus T_h) + \mu_h^2(\mathcal{T}_h \cap \mathcal{T}_H) \\ &\leq \frac{3}{4}(1 + \delta_1)\mu_H^2(\mathcal{T}_H \setminus T_h) + (1 + \delta_1)\mu_H^2(\mathcal{T}_h \cap \mathcal{T}_H) + \Lambda_1 H^2 E_H^2. \\ &\leq (1 + \delta_1)\mu_H^2 - \frac{1}{4}(1 + \delta_1)\mu_H^2(\mathcal{T}_H \setminus T_h) + \Lambda_1 H^2 E_H^2 \\ &\leq (1 + \delta_1)(1 - \frac{\theta_B}{4})\mu_H^2 + \Lambda_1 H^2 E_H^2. \end{aligned}$$

For given $0 < \theta_B < 1$, the selection $0 < \delta_1 < \theta_B/(4 - \theta_B)$ results in $0 < \rho_B = (1 + \delta_1)(1 - \frac{\theta_B}{4}) < 1$. This concludes the proof. \square

Remark 3.1. *Irrespective of the marking criteria, from (3.26)-(3.27), it follows that*

$$\mu_h^2 \leq 2\mu_H^2 + \Lambda_1 H^2 E_H^2.$$

COROLLARY 3.5. *Let \mathcal{T}_h be a refined triangulation of \mathcal{T}_H . Then it holds,*

$$(3.32) \quad \frac{\gamma_0^2}{2} \mu_H^2 \leq \mu_h^2 + 2\|H(f_h - f_H)\|^2 + C_1 H^2 E_H^2,$$

where for $T \in \mathcal{T}_h$, $T' \in \mathcal{T}_H$ and $T \subset T'$, γ_0 is defined as $0 < \gamma_0 < 1$ if T' get refined, otherwise $\gamma_0 = 1$.

LEMMA 3.6. (*Edge estimator reduction*) *Given $0 < \theta_A \leq 1$, there exist constants $0 < \delta_2$, $\rho_A < 1$ and a positive constant Λ_2 such η_H and E_H defined in (2.4) and (3.25) satisfy*

$$(3.33) \quad \eta_h^2 \leq \rho_A \eta_H^2 + \Lambda_2 E_H^2 \quad \text{for the } \mathbf{Case (A)},$$

$$(3.34) \quad \eta_h^2 \leq (1 + \delta_2)\eta_H^2 + \Lambda_2 E_H^2 \quad \text{for the } \mathbf{Case (B)}.$$

Proof. For all $F \in \mathcal{E}_H$, either $F \in \mathcal{E}_h$ or there exist $E_1, E_2, \dots, E_J \in \mathcal{E}_h$ with $F = E_1 \cup E_2 \cup \dots \cup E_J$ for $J \geq 2$. For the case $F \in \mathcal{E}_H \setminus \mathcal{E}_h$, a use of Young's inequality yields

$$\begin{aligned} \eta_h^2(F) &= \sum_{j=1}^J \eta_h^2(E_j) = \sum_{j=1}^J \|h_{E_j}^{1/2}[\mathbf{A}^{-1}\mathbf{p}_h + u_h \mathbf{b}^*] \cdot \tau_{E_j}\|_{E_j}^2 \\ &\leq (1 + \delta_2) \frac{H_F}{2} \|[\mathbf{A}^{-1}\mathbf{p}_h + u_h \mathbf{b}^*] \cdot \tau_E\|_F^2 \\ &\quad + (1 + \frac{1}{\delta_2}) \sum_{j=1}^J h_{E_j} \|[\mathbf{A}^{-1}(\mathbf{p}_h - \mathbf{p}_H) + (u_h - u_H)\mathbf{b}^*] \cdot \tau_{E_j}\|_{E_j}^2. \end{aligned} \quad (3.35)$$

Here H_F denotes the length of edge F . The fact that $J \geq 2$ implies that there exists at least one bisection of $F \in \mathcal{E}_H \setminus \mathcal{E}_h$.

For $F \in \mathcal{E}_H \cap \mathcal{E}_h$,

$$(3.36) \quad \begin{aligned} \eta_h^2(F) &\leq (1 + \delta_2) \|H_F [\mathbf{A}^{-1} \mathbf{p}_H + u_H \mathbf{b}^*] \cdot \tau_E\|_E^2 \\ &\quad + (1 + \frac{1}{\delta_2}) H_F \|[\mathbf{A}^{-1}(\mathbf{p}_h - \mathbf{p}_H) + (u_h - u_H) \mathbf{b}^*] \cdot \tau_F\|_F^2. \end{aligned}$$

For any $E \in \mathcal{E}_h$ with $E \not\subseteq \cup \mathcal{E}_H$, E is the interior edge of some element of $T \in \mathcal{T}_H$ and hence, $[\mathbf{A}^{-1} \mathbf{p}_H + u_H \mathbf{b}^*] \cdot \tau_E = 0$ along E implies $\eta_h^2(E) = \|h_E [\mathbf{A}^{-1}(\mathbf{p}_h - \mathbf{p}_H) + (u_h - u_H) \mathbf{b}^*] \cdot \tau_E\|_E^2$. Now, consider this and the cases (3.35)-(3.36) to obtain

$$\begin{aligned} \eta_h^2 &= \sum_{E \in \mathcal{E}_h} \eta_h^2(E) \leq \frac{1 + \delta_2}{2} \sum_{E \in \mathcal{E}_H \setminus \mathcal{E}_h} \eta_H^2(E) + (1 + \delta_2) \sum_{E \in \mathcal{E}_H \cap \mathcal{E}_h} \eta_H^2(E) \\ &\quad + (1 + \frac{1}{\delta_2}) \sum_{E \in \mathcal{E}_h} \|h_E [\mathbf{A}^{-1}(\mathbf{p}_h - \mathbf{p}_H) + (u_h - u_H) \mathbf{b}^*] \cdot \tau_E\|_E^2. \end{aligned}$$

The inverse inequality $\|\mathbf{q}_H\|_E^2 \lesssim H^{-\frac{1}{2}} \|\mathbf{q}_H\|_{\omega_E}^2$ results in

$$\|h_E [\mathbf{A}^{-1}(\mathbf{p}_h - \mathbf{p}_H) + (u_h - u_H) \mathbf{b}^*] \cdot \tau_E\|_E^2 \lesssim \|(\mathbf{A}^{-1}(\mathbf{p}_h - \mathbf{p}_H) + (u_h - u_H) \mathbf{b}^*)\|_{\omega_E}^2$$

for the edge patch ω_E of E in \mathcal{T}_h . Since there is only a finite overlap of all edge patches, there holds

$$\eta_h^2 \leq \frac{1 + \delta_2}{2} \sum_{E \in \mathcal{E}_H \setminus \mathcal{E}_h} \eta_H^2 + (1 + \delta_2) \sum_{E \in \mathcal{E}_H \cap \mathcal{E}_h} \eta_H^2 + C_2(1 + \frac{1}{\delta_2}) E_H^2.$$

Denote $C_2(1 + \frac{1}{\delta_2}) =: \Lambda_2$ to obtain

$$(3.37) \quad \eta_h^2 \leq \frac{1 + \delta_2}{2} \sum_{E \in \mathcal{E}_H \setminus \mathcal{E}_h} \eta_H^2 + (1 + \delta_2) \sum_{E \in \mathcal{E}_H \cap \mathcal{E}_h} \eta_H^2 + \Lambda_2 E_H^2.$$

For the **Case (A)**, the marking criteria for $\mathcal{M}_H \in \mathcal{E}_H$,

$$\theta_A \eta_H^2 \leq \eta_H(\mathcal{M}_H) \leq \eta(\mathcal{E}_H \setminus \mathcal{E}_h)$$

leads to

$$\begin{aligned} \eta_h^2 &\leq (1 + \delta_2) \eta_H^2 - \frac{(1 + \delta_2)}{2} \sum_{E \in \mathcal{E}_H \setminus \mathcal{E}_h} \eta_H^2 + \Lambda_2 E_H^2 \\ &\leq (1 + \delta_2) (1 - \frac{\theta_A}{2}) \eta_H^2 + \Lambda_2 E_H^2. \end{aligned}$$

For any given θ_A , the choice of $\delta_2 = \theta_A / (4 - 2\theta_A)$ implies $0 < \rho_A = (1 + \delta_2)(1 - \frac{\theta_A}{4}) < 1$. Hence, (3.33) holds. For the **Case (B)**, a summation over the all edges implies

$$\eta_h^2 \leq (1 + \delta_2) \eta_H^2 + \Lambda_2 E_H^2.$$

This completes the rest of the proof. \square

LEMMA 3.7. (*Quasi-orthogonality*) Let \mathcal{T}_h be a refined triangulation of \mathcal{T}_H . Then for small initial mesh-size $h \leq h_2$, there exist constants $0 < \alpha_1, \alpha_3 < 1$ such that

$$(3.38) \quad (1 - \alpha_1)e_h^2 \leq e_H^2 - \alpha_3 E_H^2 + \Lambda_3 \mu_H^2.$$

Proof. The following hold

$$(3.39) \quad \begin{aligned} \|\mathbf{A}^{-1/2}(\mathbf{p} - \mathbf{p}_h)\|^2 &= \|\mathbf{A}^{-1/2}(\mathbf{p} - \mathbf{p}_H)\|^2 - \|\mathbf{A}^{-1/2}(\mathbf{p}_h - \mathbf{p}_H)\|^2 \\ &\quad - 2(\mathbf{A}^{-1}(\mathbf{p} - \mathbf{p}_h), \mathbf{p}_h - \mathbf{p}_H), \end{aligned}$$

$$(3.40) \quad \|u - u_h\|^2 = \|u - u_H\|^2 - \|u_h - u_H\|^2 - 2(u - u_h, u_h - u_H).$$

The term $(\mathbf{A}^{-1}(\mathbf{p} - \mathbf{p}_h), \mathbf{p}_h - \mathbf{p}_H)$ is estimated by introducing the intermediate term $\tilde{\mathbf{p}} = -(\mathbf{A}\nabla_{NC}u_h^N + u_h\mathbf{b})$ as

$$(3.41) \quad \begin{aligned} (\mathbf{A}^{-1}(\mathbf{p} - \mathbf{p}_h), \mathbf{p}_h - \mathbf{p}_H) &= (\mathbf{A}^{-1}(\mathbf{p} - \tilde{\mathbf{p}}), \mathbf{p}_h - \mathbf{p}_H) + (\mathbf{A}^{-1}(\tilde{\mathbf{p}} - \mathbf{p}_h), \mathbf{p}_h - \mathbf{p}_H) \\ &= -(\nabla_{NC}(u - u_h^N), \mathbf{p}_h - \mathbf{p}_H) - ((u - u_h)\mathbf{b}^*, \mathbf{p}_h - \mathbf{p}_H) \\ &\quad + (\mathbf{A}^{-1}(\tilde{\mathbf{p}} - \mathbf{p}_h), \mathbf{p}_h - \mathbf{p}_H). \end{aligned}$$

An elementwise integration by parts of the first term on the right-hand side of (3.41) yields

$$(3.42) \quad \begin{aligned} -(\nabla_{NC}(u - u_h^N), \mathbf{p}_h - \mathbf{p}_H) &= (u - u_h^N, \operatorname{div}(\mathbf{p}_h - \mathbf{p}_H)) \\ &\quad - \sum_{E \in \mathcal{E}_h} \int_E [u - u_h^N](\mathbf{p}_h - \mathbf{p}_H) \cdot \nu_E \, ds. \end{aligned}$$

The second term on right-hand side of (3.42) is zero, as $(\mathbf{p}_h - \mathbf{p}_H) \cdot \nu_E$ is continuous along the edge E and constant on E , and $u_h^N \in CR_0^1(\mathcal{T})$, $\int_E [u - u_h^N] ds = 0$. The second equation of weak formulation (2.2) and the definition of L^2 -projection show

$$(3.43) \quad \begin{aligned} (u - u_h^N, \operatorname{div}(\mathbf{p}_h - \mathbf{p}_H)) \\ = ((u - u_h^N) - \Pi_H(u - u_h^N), f_h - f_H) - (u - u_h^N, \gamma(u_h - u_H)). \end{aligned}$$

A substitution of (3.42)-(3.43) in (3.41) with a use of the Cauchy-Schwarz inequality and the Poincaré inequality results in

$$\begin{aligned} (\mathbf{A}^{-1}(\mathbf{p} - \mathbf{p}_h), \mathbf{p}_h - \mathbf{p}_H) &\lesssim \|\nabla_{NC}(u - u_h^N)\| \|H(f_h - f_H)\| + \|u - u_h^N\| \|u_h - u_H\| \\ &\quad + \|u - u_h\| \|\mathbf{A}^{-1/2}(\mathbf{p}_h - \mathbf{p}_H)\| + \|\tilde{\mathbf{p}} - \mathbf{p}_h\| \|\mathbf{A}^{-1/2}(\mathbf{p}_h - \mathbf{p}_H)\|. \end{aligned}$$

Using the estimates (3.16)-(3.21), Lemma 3.1 and the addition of the term $(u - u_h, u_h - u_H)$ yield

$$\begin{aligned} 2|(\mathbf{A}^{-1}(\mathbf{p} - \mathbf{p}_h), \mathbf{p}_h - \mathbf{p}_H)| + 2|(u - u_h, u_h - u_H)| &\leq C_3(e_{p_h} + e_{u_h}) \operatorname{osc}_H(f) \\ &\quad + C_3(\|hf_h\| + \|h(\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*)\|) \operatorname{osc}_H(f) + C_3\epsilon(e_{p_h} + e_{u_h})(\|u_h - u_H\| \\ &\quad + \|\mathbf{A}^{-1/2}(\mathbf{p}_h - \mathbf{p}_H)\|) + C_3(\operatorname{osc}_h(f) + \|hf_h\| + \|h\mathbf{A}^{-1}\mathbf{p}_h + u_h\mathbf{b}^*\|) \\ &\quad (\|u_h - u_H\| + \|\mathbf{A}^{-1/2}(\mathbf{p}_h - \mathbf{p}_H)\|). \end{aligned}$$

The Young's inequality and rearrangement of terms result in

$$(3.44) \quad \begin{aligned} 2|(\mathbf{A}^{-1}(\mathbf{p} - \mathbf{p}_h), \mathbf{p}_h - \mathbf{p}_H)| + 2|(u - u_h, u_h - u_H)| \\ \leq (\delta_3 + 4C_3^2\epsilon^2)(e_{p_h}^2 + e_{u_h}^2) + \frac{1}{2}E_H^2 + 7C_3^2\mu_h^2 + \left(\frac{1}{2} + \frac{C_3^2}{2\delta_3}\right) \operatorname{osc}_H^2(f). \end{aligned}$$

As \mathcal{T}_h and \mathcal{T}_H are the nested triangulations, Remark 3.1 implies that $\mu_h^2 \leq 2\mu_H^2 + \Lambda_1 H^2 E_H^2$. Hence, with notations $0 < \alpha_1 = (\delta_3 + 4C_3^2 \epsilon^2)$, $\alpha_2 = (\frac{1}{2} + 7C_3^2 \Lambda_1 H^2)$ and $\Lambda_3 = (14C_3^2 + \frac{1}{2} + \frac{C_3^2}{2\delta_3})$, (3.44) reduces to

$$(3.45) \quad \begin{aligned} & 2|(\mathbf{A}^{-1}(\mathbf{p} - \mathbf{p}_h), \mathbf{p}_h - \mathbf{p}_H)| + 2|(u - u_h, u_h - u_H)| \\ & \leq \alpha_1(e_{p_h}^2 + e_{u_h}^2) + \alpha_2 E_H^2 + \Lambda_3 \mu_H^2. \end{aligned}$$

A combination of (3.39)-(3.40) with (3.45) yields

$$(e_{p_h}^2 + e_{u_h}^2) \leq (e_{p_H}^2 + e_{u_H}^2) - E_H^2 + \alpha_1(e_{p_h}^2 + e_{u_h}^2) + \alpha_2 E_H^2 + \Lambda_3 \mu_H^2.$$

Hence $(1 - \alpha_1)e_h^2 \leq e_H^2 - (1 - \alpha_2)E_H^2 + \Lambda_3 \mu_H^2$.

For any $\epsilon > 0$, it is always possible to find a small initial mesh-size h_2 and $0 < \delta_3 < 1$, such that $0 < \alpha_1 = \delta_3 + 4C_3^2 \epsilon^2 < 1$ and $0 < \alpha_3 = 1 - \alpha_2 = 1/2 - 7C_3^2 \Lambda_1 H^2$. Hence, (3.38) holds true and this completes the rest of the proof. \square

Remark 3.2. For proving the contraction property, a more sharper bound for $\alpha_1 < \alpha_4 := 1/2 \min\{1, \alpha_3(1 - \rho_A)/((\Lambda_1 + \Lambda_2)C_{rel})\} < 1$ will be selected.

COROLLARY 3.8. Under the assumption that small initial mesh-size $h_2 > 0$, and the constants defined in Lemma 3.7, the following result holds for $0 < h \leq h_2$

$$(3.46) \quad \begin{aligned} \mathbf{e}_H^2 & \leq (1 + \alpha_1)\mathbf{e}_h^2 + (1 + \alpha_2)E_H^2 + \Lambda_3 \mu_H^2 \\ & \leq 2\mathbf{e}_h^2 + 2E_H^2 + \Lambda_3 \mu_H^2. \end{aligned}$$

LEMMA 3.9. (Quasi-discrete reliability) Let (\mathbf{p}_h, u_h) and (\mathbf{p}_H, u_H) be the MFEM solutions of (2.1)-(2.2) over the triangulations \mathcal{T}_h and \mathcal{T}_H , respectively. There exists a constant $C_4 > 0$ such that for any $\epsilon > 0$

$$\begin{aligned} \|\mathbf{A}^{-1/2}(\mathbf{p}_h - \mathbf{p}_H)\|^2 + \|u_h - u_H\|^2 & \leq C_4(\eta_H^2(\mathcal{E}_H \setminus \mathcal{E}_h) + \|H(f_h - f_H)\|^2 \\ & \quad + \epsilon^2(e_h^2 + e_H^2) + \mu_h^2 + \mu_H^2), \end{aligned}$$

where E_H, e_h, e_H are defined in (3.25) and μ_h in (2.5).

Proof. Introduce the discrete mixed finite element problem: seek $(\tilde{\mathbf{p}}_h, \tilde{u}_h) \in RT_0(\mathcal{T}_h) \times P_0(\mathcal{T}_h)$ such that

$$(3.47) \quad (\mathbf{A}^{-1}\tilde{\mathbf{p}}_h, \mathbf{q}_h) - (\operatorname{div} \mathbf{q}_h, \tilde{u}_h) = -(u_H \mathbf{b}^*, \mathbf{q}_h) \quad \text{for all } \mathbf{q}_h \in RT_0(\mathcal{T}_h),$$

$$(3.48) \quad (\operatorname{div} \tilde{\mathbf{p}}_h, v_h) = (f_H, v_h) - (\gamma u_H, v_h) \quad \text{for all } v_h \in P_0(\mathcal{T}_h).$$

Define the nonconforming discrete problem corresponding to (3.47)-(3.48): seek $\tilde{u}_h^N \in CR_0^1(\mathcal{T}_h)$ as the solution of

$$(3.49) \quad (\mathbf{A} \nabla_{NC} \tilde{u}_h^N + u_H \mathbf{b}, \nabla_{NC} v_{CR}) + (\gamma u_H, v_{CR}) = (f_H, v_{CR}), \forall v_{CR} \in CR_0^1(\mathcal{T}_h).$$

The solution $\tilde{\mathbf{p}}_h$ of (3.47)-(3.48) can be written in the terms of \tilde{u}_h^N as

$$(3.50) \quad \tilde{\mathbf{p}}_h(\mathbf{x}) = -(\mathbf{A} \nabla_{NC} \tilde{u}_h^N + u_H \mathbf{b}) + (f_H - \gamma u_H) \frac{(\mathbf{x} - \operatorname{mid}(T))}{2} \quad \text{for } \mathbf{x} \in T.$$

Now for the estimates of $\|\mathbf{p}_h - \mathbf{p}_H\|$, use $\tilde{\mathbf{p}}_h$ as an intermediate term to split $\mathbf{p}_h - \mathbf{p}_H := (\mathbf{p}_h - \tilde{\mathbf{p}}_h) + (\tilde{\mathbf{p}}_h - \mathbf{p}_H)$ and the triangle inequality. From the representation formula (3.5) and (3.50) of \mathbf{p}_h and $\tilde{\mathbf{p}}_h$, respectively, it follows that

$$\mathbf{p}_h - \tilde{\mathbf{p}}_h = -(\mathbf{A} \nabla(u_h^N - \tilde{u}_h^N) + (u_h - u_H)\mathbf{b}) + \frac{1}{2}(f_h - f_H - \gamma(u_h - u_H))(x - \operatorname{mid}(T)).$$

The triangle inequality shows

$$(3.51) \quad \|\mathbf{p}_h - \tilde{\mathbf{p}}_h\| \lesssim \|\mathbf{A}\nabla(u_h^N - \tilde{u}_h^N)\| + \|u_h - u_H\| + \|h(f_h - f_H)\|.$$

Subtracting (3.2) from (3.49) leads to

$$(3.52) \quad \begin{aligned} (\mathbf{A}\nabla(u_h^N - \tilde{u}_h^N), \nabla v_{CR}) &= (f_h - f_H, v_{CR}) + (\gamma(u_h - u_H), v_{CR}) \\ &\quad - ((u_h - u_H)\mathbf{b}, \nabla v_{CR}). \end{aligned}$$

A substitution $v_{CR} = \mathbf{A}(u_h^N - \tilde{u}_h^N)$ in (3.52) with the L^2 -projection property and the Cauchy-Schwarz inequality results in

$$\|\mathbf{A}\nabla(u_h^N - \tilde{u}_h^N)\| \lesssim \|H(f_h - f_H)\| + \|u_h - u_H\|.$$

With this estimate, (3.51) reduces to

$$(3.53) \quad \|\mathbf{p}_h - \tilde{\mathbf{p}}_h\| \lesssim \|H(f_h - f_H)\| + \|u_h - u_H\|.$$

For a bound of the term $(u_h - u_H)$, a use of (3.21) shows

$$(3.54) \quad \begin{aligned} \|u_h - u_H\|^2 &\lesssim \|u_h - u\|^2 + \|u - u_H\|^2 \\ &\lesssim \epsilon^2 e_h^2 + \mu_h^2 + \epsilon^2 e_H^2 + \mu_H^2. \end{aligned}$$

For the estimate of $\|\mathbf{p}_h - \mathbf{p}_H\|$, note that $\operatorname{div} \tilde{\mathbf{p}}_h = f_h - \gamma u_H = \operatorname{div} \mathbf{p}_H$. It implies $\operatorname{div} (\tilde{\mathbf{p}}_h - \mathbf{p}_H) = 0$, and hence, $\tilde{\mathbf{p}}_h - \mathbf{p}_H$ is a piecewise constant vector function over \mathcal{T}_h . The discrete Helmholtz decomposition states for $\mathbf{A}(\tilde{\mathbf{p}}_h - \mathbf{p}_H) = \mathbf{A}\nabla\alpha_{CR} + \operatorname{Curl} \beta_h$, where $\alpha_{CR} \in CR_0^1(\mathcal{T}_h)$ and $\beta_h \in P_1(\mathcal{T}_h) \cap C(\bar{\Omega})$, and hence,

$$\|\tilde{\mathbf{p}}_h - \mathbf{p}_H\|^2 = (\tilde{\mathbf{p}}_h - \mathbf{p}_H, \nabla\alpha_{CR} + \mathbf{A}^{-1}\operatorname{Curl} \beta_h).$$

Let $\beta_H := I_H\beta_h$ be the Scott-Zhang quasi-interpolation operator for $E \in \mathcal{E}_H$, where \mathcal{E}_H the set of edges on the triangulation \mathcal{T}_H and its neighbourhood ω_E with

$$(3.55) \quad \|\beta_h - \beta_H\|_E \leq Ch_E^{1/2} \|\beta_h\|_{H^1(\omega_E)}.$$

Note that $\|\beta_h - \beta_H\|_E = 0$ if $E \in \mathcal{E}_h \cap \mathcal{E}_H$.

The fact $((\tilde{\mathbf{p}}_h - \mathbf{p}_H), \nabla_{NC}\alpha_{CR}) = 0$ shows $\|\tilde{\mathbf{p}}_h - \mathbf{p}_H\|^2 = (\tilde{\mathbf{p}}_h - \mathbf{p}_H, \mathbf{A}^{-1}\operatorname{Curl} \beta_h)$. The weak formulation (2.1) with $\mathbf{q}_H = \operatorname{Curl} \beta_H \in RT_0(\mathcal{T}_H) \subset RT_0(\mathcal{T}_h)$ over \mathcal{T}_H and (3.47) with $\mathbf{q}_h = \operatorname{Curl} \beta_h \in RT_0(\mathcal{T}_h)$ and integration by parts lead to

$$\begin{aligned} \|\tilde{\mathbf{p}}_h - \mathbf{p}_H\|^2 &= -(u_H \mathbf{b}^*, \operatorname{Curl} \beta_h) - (\mathbf{A}^{-1} \mathbf{p}_H, \operatorname{Curl} \beta_h) \\ &= (\mathbf{A}^{-1} \mathbf{p}_H + u_H \mathbf{b}^*, \operatorname{Curl} (\beta_H - \beta_h)) \\ &= \sum_{E \in \mathcal{E}_h} \int_E [\mathbf{A}^{-1} \mathbf{p}_H + \mathbf{b}^* u_H] \cdot \tau_E (\beta_H - \beta_h) \, ds \\ &\quad - \sum_{T \in \mathcal{T}_h} \int_T \operatorname{Curl} (\mathbf{A}^{-1} \mathbf{p}_H + u_H \mathbf{b}^*) (\beta_H - \beta_h) \, ds. \end{aligned}$$

Since $\operatorname{Curl} (\mathbf{A}^{-1} \mathbf{p}_H + u_H \mathbf{b}^*) = 0$

$$(3.56) \quad \|\tilde{\mathbf{p}}_h - \mathbf{p}_H\|^2 \lesssim \sum_{E \in \mathcal{E}_h \setminus \mathcal{E}_H} \|[\mathbf{A}^{-1} \mathbf{p}_H + u_H \mathbf{b}^*] \cdot \tau_E\|_E \|\beta_h - \beta_H\|_E.$$

The estimates of $\|\beta_h - \beta_H\|_E$ from (3.55) and the bound $\|\nabla \beta_h\| = \|\text{Curl } \beta_h\| \lesssim \|\tilde{\mathbf{p}}_h - \mathbf{p}_H\|$ together with (3.56) result in

$$(3.57) \quad \|\tilde{\mathbf{p}}_h - \mathbf{p}_H\| \lesssim \eta_H(\mathcal{E}_H \setminus \mathcal{E}_h).$$

A combination of (3.57), (3.53) and (3.54) leads to

$$(3.58) \quad \|\mathbf{p}_h - \mathbf{p}_H\|^2 \lesssim \eta_H^2(\mathcal{E}_H \setminus \mathcal{E}_h) + \|H(f_h - f_H)\|^2 + \epsilon^2(e_h^2 + e_H^2) + \mu_h^2 + \mu_H^2.$$

The combination of (3.54) and (3.58) completes the proof. \square

4. Convergence Analysis. This section is devoted to the convergence analysis of the adaptive mixed finite element method.

4.1. Contraction property. Denote two consecutive adaptive loop levels as ℓ and $\ell + 1$. Let e_ℓ , η_ℓ and μ_ℓ denote the error and the estimator terms on the level ℓ with triangulation \mathcal{T}_ℓ . Based on the reduction properties of the error, the error estimators and quasi-orthogonal property developed in last section, the contraction property is proved for the weighted term ξ_ℓ^2 , which is a linear combination of error e_ℓ^2 and the estimator terms η_ℓ^2 and μ_ℓ^2 , between two consecutive adaptive loops.

THEOREM 4.1. (*Contraction Property*) *Let $\mathcal{T}_{\ell+1}$ be a refinement of \mathcal{T}_ℓ using AMFEM algorithm. Given $0 < \theta_A, \theta_B < 1$, there exist positive parameters α, β, κ and $0 < \rho < 1$ depending on constants $\alpha_1, \alpha_2, \Lambda_1, \Lambda_2, \Lambda_3$ from Lemmas 3.6, 3.4 and 3.7 such that on any level $\ell \geq 0$, the weighted term ξ_ℓ^2 satisfies the following contraction property:*

$$\xi_{\ell+1}^2 \leq \rho \xi_\ell^2, \quad \text{where } \xi_\ell^2 := \eta_\ell^2 + \alpha e_\ell^2 + \beta \mu_\ell^2,$$

whenever the initial mesh is chosen with small mesh-size.

Proof. For the **Case (A)**, $\mu_\ell^2 \leq \kappa \eta_\ell^2$. The combination of (3.26) and (3.33) with a positive parameter β , to be chosen later, yields

$$(4.1) \quad \eta_{\ell+1}^2 + \beta \mu_{\ell+1}^2 \leq \rho_A \eta_\ell^2 + (1 + \delta_1) \beta \mu_\ell^2 + (\Lambda_2 + \beta \Lambda_1 h_\ell^2) E_\ell^2,$$

where h_ℓ denotes the mesh-size at the level ℓ of the triangulation. With a choice of the initial mesh-size $h_3 \leq \min\{h_1, h_2, \frac{1}{\sqrt{\beta}}\}$, multiply (3.38) with the constant $C_5 = (\Lambda_1 + \Lambda_2)/\alpha_3$, and then add with (4.1) to obtain

$$\eta_{\ell+1}^2 + C_5(1 - \alpha_1)e_{\ell+1}^2 + \beta \mu_{\ell+1}^2 \leq C_5 e_\ell^2 + \rho_A \eta_\ell^2 + ((1 + \delta_1)\beta + C_5 \Lambda_3) \mu_\ell^2.$$

Define $\alpha := C_5(1 - \alpha_1)$; $0 < \alpha_4 := \frac{1}{2} \min\{1, \frac{1 - \rho_A}{C_5 C_{rel}}\}$, and use the reliability result (3.8) that is, $e_\ell^2 \leq C_{rel}(\eta_\ell^2 + \mu_\ell^2)$ to obtain

$$(4.2) \quad \begin{aligned} \xi_{\ell+1}^2 &\leq C_5(1 - \alpha_4)e_\ell^2 + C_5 \alpha_4 C_{rel}(\eta_\ell^2 + \mu_\ell^2) + \rho_A \eta_\ell^2 + ((1 + \delta_1)\beta + C_5 \Lambda_3) \mu_\ell^2 \\ &\leq C_5(1 - \alpha_4)e_\ell^2 + (\rho_A + C_5 \alpha_4 C_{rel}) \eta_\ell^2 + ((1 + \delta_1)\beta + C_5 \Lambda_3 + C_5 \alpha_4 C_{rel}) \mu_\ell^2. \end{aligned}$$

Since the marking criteria implies $E \kappa \eta_\ell^2 - E \mu_\ell^2 > 0$, where $E = 2(C_5 \Lambda_3 + C_5 \alpha_4 C_{rel})$, an addition of this term on the right-hand side of (4.2) yields

$$(4.3) \quad \begin{aligned} \xi_{\ell+1}^2 &\leq C_5(1 - \alpha_4)e_\ell^2 + (\rho_A + C_5 \alpha_4 C_{rel} + E \kappa) \eta_\ell^2 \\ &\quad + ((1 + \delta_1)\beta - (C_5 \Lambda_3 + C_5 \alpha_4 C_{rel})) \mu_\ell^2. \end{aligned}$$

A use of Remark 3.2 and the definition of α_4 with the choice of the parameters

$$\kappa < \kappa_0 := \frac{(1 - \rho_A - C_5 \alpha_4 C_{rel})}{E}, \quad \beta \geq 2(C_5 \Lambda_4 + C_5 \alpha_4 C_{rel}), \quad \text{and} \quad \delta_1 < \min \left\{ \frac{E}{2\beta}, \frac{\theta_B}{4 - \theta_B} \right\},$$

yield on any level $\mathcal{T}_{\ell+1}$, a contraction $\xi_{\ell+1}^2 \leq \rho_1 \xi_\ell^2$, where

$$0 < \rho_1 = \max \left\{ \frac{C_5(1 - \alpha_4)}{\alpha}, \rho_A + C_5 \alpha_4 C_{rel} + E\kappa, \frac{((1 + \delta_1)\beta - E/2)}{\beta} \right\} < 1.$$

For the **Case (B)**: $\mu_\ell^2 > \kappa \eta_\ell^2$. Similar to proof of the **Case (A)**, the equation corresponding to (4.2) for the **Case (B)** is

$$\begin{aligned} \xi_{\ell+1}^2 &\leq C_5(1 - \alpha_4)e_\ell^2 + C_5 \alpha_4 C_{rel}(\eta_\ell^2 + \mu_\ell^2) + (1 + \delta_2)\eta_\ell^2 + (\rho_B \beta + C_5 \Lambda_3)\mu_\ell^2 \\ (4.4) \quad &\leq C_5(1 - \alpha_4)e_\ell^2 + (1 + \delta_2 + C_5 \alpha_4 C_{rel})\eta_\ell^2 + (\rho_B \beta + C_5 \Lambda_3 + C_5 \alpha_4 C_{rel})\mu_\ell^2. \end{aligned}$$

The marking criteria implies $D\mu_\ell^2 - D\kappa\eta_\ell^2 > 0$, where $0 < D = \frac{1.5 + C_5 \alpha_4 C_{rel}}{\kappa}$. Add this term on the right-hand side of (4.4) to obtain

$$\xi_{\ell+1}^2 \leq C_5(1 - \alpha_4)e_\ell^2 + (2 + C_5 \alpha_4 C_{rel} - D\kappa)\eta_\ell^2 + (\rho_B \beta + C_5 \Lambda_3 + C_5 \alpha_4 C_{rel} + D)\mu_\ell^2.$$

A use of Remark 3.2, the parameters choice and $\beta > \frac{C_5 \Lambda_4 + C_5 \alpha_4 C_{rel} + D}{1 - \rho_B}$ yields that, on any level \mathcal{T}_ℓ the contraction $\xi_{\ell+1}^2 \leq \rho_2 \xi_\ell^2$ holds true, where ρ_2 is defined by

$$0 < \rho_2 = \max \left\{ \frac{C_5(1 - \alpha_4)}{\alpha}, 2 + C_5 \alpha_4 C_{rel} - D\kappa, \frac{\rho_B \beta + C_5 \Lambda_3 + C_5 \alpha_4 C_{rel} + D}{\beta} \right\} < 1.$$

Finally, the combination of both cases with $0 < \beta = 2 \max \left\{ C_5 \Lambda_3 + C_5 \alpha_4 C_{rel}, \frac{C_5 \Lambda_3 + C_5 \alpha_4 C_{rel} + D}{1 - \rho_B} \right\}$, $\rho = \max\{\rho_1, \rho_2\}$ and the initial mesh-size h_3 implies that (4.1) holds. \square

THEOREM 4.2. (*Convergence*) *Under the assumptions of Theorem 4.1, there exist a constant $\rho \in (0, 1)$ and $C_0 > 0$ depending only on the given data and the initial triangulation such that*

$$\eta_\ell^2 + \alpha e_\ell^2 + \beta \mu_\ell^2 \leq C_0 \rho^\ell.$$

Proof. The proof is a consequence of the contraction property in Theorem 4.1. \square

Remark 4.1. *Using the reliability result (3.8) and the relation of the error estimator η_ℓ^2 and μ_ℓ^2 in the marking strategy, the weighted term ξ_ℓ , the error term e_ℓ and the error estimator terms η_ℓ and μ_ℓ over triangulation \mathcal{T}_ℓ are obtained. Now ξ_ℓ satisfies*

$$\xi_\ell^2 \leq \begin{cases} C_6 \eta_\ell^2 & \text{for the Case (A),} \\ \frac{C_6}{\kappa} \mu_\ell^2 & \text{for the Case (B),} \end{cases}$$

where $C_6 = 1 + \alpha C_{rel} + (\alpha C_{rel} + \beta)\kappa$. The reliability (3.8), Lemma 3.1 with the efficiency result (3.22) implies

$$\xi_\ell^2 \approx e_\ell^2 + \|h_{\mathcal{T}} f\|^2.$$

LEMMA 4.3. *Let \mathcal{T}_h and \mathcal{T}_H be two nested triangulations. Then for small mesh-size $h_4 > 0$, it holds for $0 < h \leq h_4$*

$$(4.5) \quad \xi_h^2 \lesssim \xi_H^2.$$

Proof. From Lemma 3.7, Remark 3.1 and the efficiency result (3.3) implies

$$(4.6) \quad \begin{aligned} \xi_h^2 &= \eta_h^2 + \alpha e_h^2 + \beta \mu_h^2 \leq (C_{\text{eff}}^{-1} + \alpha) e_h^2 + 2\beta \mu_H^2 + \beta \Lambda_1 H^2 E_H^2 \\ &\leq \frac{(C_{\text{eff}}^{-1} + \alpha)}{1 - \alpha_1} (e_H^2 - (\frac{1}{2} - 8C_3 \Lambda_1 H^2) E_H^2 + \Lambda_3 \mu_H^2) + 2\beta \mu_H^2 + \beta \Lambda_1 H^2 E_H^2. \end{aligned}$$

Select the initial mesh-size $h_4 > 0$ such that the coefficient of E_H^2 is non-positive for $0 < h \leq h_4$. Then (4.6) implies (4.5). Note that the inequality constant in (4.5) is independent of AMFEM marking parameters. This concludes the proof. \square

4.2. Quasi-optimality. In this subsection, the quasi-optimal convergence [33] of the adaptive algorithm MFEM is discussed with the help of the quasi-discrete reliability and the contraction property.

DEFINITION 4.4. [8, 33] (*Approximation class*) Given an initial triangulation \mathcal{T}_0 of Ω and $s > 0$, the approximation class is defined as

$$\mathcal{A}_s := \{(\mathbf{p}, u, f) \in H(\text{div}, \Omega) \times L^2(\Omega) \times L^2(\Omega) \mid \|(\mathbf{p}, u, f)\|_{\mathcal{A}_s} < \infty\} \quad \text{with}$$

$$\|(\mathbf{p}, u, f)\|_{\mathcal{A}_s} := \sup_{N \in \mathbb{N}} \left(N^s \inf_{|\mathcal{T}| - |\mathcal{T}_0| \leq N} (e^2(\mathcal{T}) + \|h_{\mathcal{T}} f\|^2)^{1/2} \right) \text{ and } e^2(\mathcal{T}) = e_p^2(\mathcal{T}) + e_u^2(\mathcal{T}).$$

Here, the infimum is over all regular and NVB-generated refinements \mathcal{T} of \mathcal{T}_0 with the number of element domains $|\mathcal{T}| \leq N + |\mathcal{T}_0|$ and the exact error $e(\mathcal{T})$ in MFEM solution. An adaptive mixed finite element method is quasi-optimal convergent in the sense that given $(\mathbf{p}, u, f) \in \mathcal{A}_s$, and $j \in \mathbb{N}$, the AMFEM algorithm generates a triangulation \mathcal{T}_j with discrete solution $(\mathbf{p}_j, u_j) \in RT_0(\mathcal{T}_j) \times P_0(\mathcal{T}_j)$ such that

$$|\mathcal{T}_j| - |\mathcal{T}_0| \leq \xi_j^{-\frac{1}{s}} \approx (e_j^2 + \|h_{\mathcal{T}} f\|^2)^{-\frac{1}{2s}}.$$

THEOREM 4.5. (*Quasi-optimality*) Assume $(\mathbf{p}, u, f) \in \mathcal{A}_s$. Let $\{\mathcal{T}_j\}_{j \geq 0}$ be the sequence of the meshes generated by AMFEM algorithm and $\{(\mathbf{p}_j, u_j) \in RT_0(\mathcal{T}_j) \times P_0(\mathcal{T}_j)\}_{j \geq 0}$ be a corresponding sequence of approximate solutions. Then, for small initial mesh-size h_0 , the following estimates holds true for $0 < h \leq h_0$

$$(4.7) \quad |\mathcal{T}_j| - |\mathcal{T}_0| \lesssim \xi_j^{-\frac{1}{s}}.$$

Proof. Consider

$$(4.8) \quad |\mathcal{T}_j| - |\mathcal{T}_0| \leq \sum_{\ell=0}^{j-1} (|\mathcal{T}_{\ell+1}| - |\mathcal{T}_{\ell}|)$$

Now $|\mathcal{T}_{\ell+1}| - |\mathcal{T}_{\ell}| \lesssim |\mathcal{M}_{\ell}|$ for $\ell \geq 0$ [6, 33], where $\mathcal{T}_{\ell+1}$ is a refinement of \mathcal{T}_{ℓ} and \mathcal{M}_{ℓ} denotes the set of the marked edges or elements in the triangulation level \mathcal{T}_{ℓ} . Then

$$(4.9) \quad |\mathcal{T}_j| - |\mathcal{T}_0| \lesssim \sum_{\ell=0}^{j-1} |\mathcal{M}_{\ell}|.$$

To estimate of $|\mathcal{M}_\ell|$, use the characterization of the approximate class and the overlay. If $(\mathbf{p}, u, f) \in \mathcal{A}_s$, then for $\epsilon_1 := \tau_1 \xi_\ell$, there exists some admissible triangulation \mathcal{T}_{ϵ_1} obtained as refinement of the initial triangulation \mathcal{T}_0 such that

$$(4.10) \quad \xi_{\epsilon_1}^2 = \eta_{\epsilon_1}^2 + \alpha e_{\epsilon_1}^2 + \beta \mu_{\epsilon_1}^2 \leq \epsilon_1^2 \quad \text{and} \quad |\mathcal{T}_{\epsilon_1}| - |\mathcal{T}_0| \lesssim \epsilon_1^{-\frac{1}{s}}.$$

Here, $0 < \tau_1 = \min\{\tau_2, \tau_3\}$, and τ_2 and τ_3 will be specified later.

Let $\mathcal{T}_{\ell+\epsilon_1} := \mathcal{T}_{\epsilon_1} \oplus \mathcal{T}_\ell$ be the overlay of \mathcal{T}_{ϵ_1} and \mathcal{T}_ℓ . As in [17], the number of elements of the overlay $\mathcal{T}_{\ell+\epsilon_1}$ can be bounded by

$$(4.11) \quad |\mathcal{T}_{\ell+\epsilon_1}| - |\mathcal{T}_\ell| \leq |\mathcal{T}_{\epsilon_1}| - |\mathcal{T}_0|.$$

To estimate $|\mathcal{M}_\ell|$ with the first case of Mark algorithm, that is, $\kappa \eta_\ell^2 \geq \mu_\ell^2$, first define $\mathcal{M}^\dagger := \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+\epsilon_1} \subset \mathcal{E}_\ell$ as the set of edges of \mathcal{T}_ℓ being refined in $\mathcal{T}_{\ell+\epsilon_1}$. Note that, if \mathcal{M}^\dagger satisfies the marking criteria of the **Case (A)**, that is,

$$(4.12) \quad \theta_A \eta_\ell^2 \leq \eta_\ell^2(\mathcal{M}^\dagger),$$

then, $|\mathcal{M}_\ell| \leq |\mathcal{M}^\dagger|$, where $|\mathcal{M}_\ell|$ be the set of marked edges at level ℓ .

The quasi-discrete reliability result Lemma 3.9 over the triangulations \mathcal{T}_ℓ and $\mathcal{T}_{\ell+\epsilon_1}$ with $\epsilon^2 = 1/4C_4$ and the initial mesh-size h_5 shows

$$C_4 \eta_\ell^2(\mathcal{M}^\dagger) \geq E_\ell^2 - C_4 \text{osc}_\ell^2(f) - \frac{1}{4}(e_\ell^2 + e_{\ell+\epsilon_1}^2) - C_4(\mu_\ell^2 + \mu_{\ell+\epsilon_1}^2).$$

Remark 3.1 yields $\mu_{\ell+\epsilon_1}^2 \leq 2\mu_\ell^2 + \Lambda_1 h_\ell^2 E_\ell^2$, where h_ℓ denotes the mesh-size over the triangulation \mathcal{T}_ℓ and note that $\text{osc}_\ell^2(f) \leq \mu_\ell^2$. Hence,

$$(4.13) \quad C_4 \eta_\ell^2(\mathcal{M}^\dagger) \geq (1 - C_4 \Lambda_1 h_\ell^2) E_\ell^2 - 4C_4 \mu_\ell^2 - \frac{1}{4}(e_\ell^2 + e_{\ell+\epsilon_1}^2).$$

For small initial mesh-size h_4 , Lemma 4.3, (4.10), the choice of ϵ_1 and Remark 4.1 imply

$$e_{\ell+\epsilon_1}^2 \leq \xi_{\ell+\epsilon_1}^2 \lesssim \xi_{\epsilon_1}^2 \lesssim \tau_1^2 \xi_\ell^2 \lesssim \tau_1^2 \eta_\ell^2 \leq \tau_2^2 \eta_\ell^2,$$

that is, $e_{\ell+\epsilon_1}^2 \leq C_7 \tau_2^2 \eta_\ell^2$. Thus, Corollary 3.8 supplies the lower bound for E_ℓ . With these results, (4.13) leads to

$$(4.14) \quad C_4 \eta_\ell^2(\mathcal{M}^\dagger) \geq \frac{1}{2} \left(\frac{1}{2} - C_4 \Lambda_1^2 h_\ell^2 \right) e_\ell^2 - \frac{5}{4} C_7 \tau_2^2 \eta_\ell^2 - \left(4C_4 + \frac{\Lambda_3}{2} \right) \mu_\ell^2.$$

Note that some positive terms are neglected from the right-hand side. Choose the initial mesh-size $h_6 := \min\{h_2, h_5, \frac{1}{(4C_4 \Lambda_1)^{1/2}}\}$ and the marking parameter $\theta_A = \min\{1, \frac{c_{\text{eff}}}{9C_4}\}$. The **Case (A)** relation $\kappa \eta_\ell^2 \geq \mu_\ell^2$ along-with the efficiency result (3.22) leads to

$$C_4 \eta_\ell^2(\mathcal{M}^\dagger) \geq \left(\frac{C_{\text{eff}}}{8} - \frac{5}{4} C_7 \tau_2^2 - \kappa \left(4C_4 + \frac{\Lambda_3}{2} \right) - C_4 \theta_A \right) \eta_\ell^2 + C_4 \theta_A \eta_\ell^2.$$

The selection of $\kappa < \min\{\kappa_0, \frac{C_{\text{eff}}/8 - C_4 \theta_A}{4C_4 + \Lambda_3/2}\}$, and $\tau_2^2 := \frac{(c_{\text{eff}}/8) - \kappa(4C_4 + \Lambda_3/2) - C_4 \theta_A}{(5/4)C_7}$ imply that (4.12) holds.

Since \mathcal{M}_ℓ is chosen to be the minimal cardinality set satisfying (4.12), Lemma 4.4 of [8], (4.11) and (4.10) altogether imply

$$(4.15) \quad |\mathcal{M}_\ell| \leq |\mathcal{M}_\ell^\dagger| \lesssim |\mathcal{T}_{\ell+\epsilon_1}| - |\mathcal{T}_\ell| \leq |\mathcal{T}_{\epsilon_1}| - |\mathcal{T}_0| \lesssim \epsilon_1^{-\frac{1}{s}} \lesssim \xi_\ell^{-\frac{1}{s}}.$$

Consider the **Case (B)** of Mark algorithm, that is, $\kappa\eta_\ell^2 \leq \mu_\ell^2$. Let $\mathcal{M}^* = \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+\epsilon_1} \subset \mathcal{T}_\ell$ be the set of elements of \mathcal{T}_ℓ refined in $\mathcal{T}_{\ell+\epsilon_1}$. The proof of \mathcal{M}^* satisfies the marking criteria of the **Case (B)**, that is,

$$(4.16) \quad \theta_B \eta_\ell^2 \leq \eta_\ell^2(\mathcal{M}^*),$$

and this will imply $|\mathcal{M}_\ell| \leq |\mathcal{M}^*|$, where \mathcal{M}_ℓ is the set of the marked elements at the level ℓ .

Corollary 3.5 and the quasi-discrete reliability result in Lemma 3.9, with $\epsilon = 1$ and initial mesh-size h_7 over the nested triangulations $\mathcal{T}_{\ell+\epsilon_1}$ and \mathcal{T}_ℓ imply

$$\begin{aligned} \mu_{\ell+\epsilon_1}^2 &\geq \frac{\gamma_0^2}{2} \mu_\ell^2 - 2\|h_\ell(f_\ell - f_{\ell+\epsilon_1})\|^2 - C_1 h_\ell^2 E_\ell^2 \\ &\geq \left(\frac{\gamma_0^2}{2} - C_4 C_1 h_\ell^2\right) \mu_\ell^2 - (2 + C_4 C_1 h_\ell^2) \|h_\ell(f_\ell - f_{\ell+\epsilon_1})\|^2 \\ &\quad - C_1 C_4 h_\ell^2 (\eta_\ell^2 + (e_\ell^2 + e_{\ell+\epsilon_1}^2) + \mu_{\ell+\epsilon_1}^2). \end{aligned}$$

The reliability result, the relation $\kappa\eta_\ell^2 < \mu_\ell^2$ for the **Case (B)** and the rearrangement of terms imply

$$(4.17) \quad \mu_{\ell+\epsilon_1}^2 \geq \frac{(\frac{\gamma_0^2}{2} - C_8 h_\ell^2)}{(2 + C_4 C_1 h_\ell^2)} \mu_\ell^2 - \|h_\ell(f_\ell - f_{\ell+\epsilon_1})\|^2 - C_1 C_4 h_\ell^2 e_{\ell+\epsilon_1}^2,$$

where $C_8 := C_4 C_1 (C_{rel} + 1)(1 + 1/\kappa)$. Lemma 4.3, the choice of ϵ_1 , that is, $\epsilon_1 = \tau_1 \xi_\ell$ and the marking criteria $\kappa\eta_\ell^2 < \mu_\ell^2$ in Remark 4.1 result in

$$(4.18) \quad \alpha e_{\ell+\epsilon_1}^2 + \beta \mu_{\ell+\epsilon_1}^2 \leq \xi_{\ell+\epsilon_1}^2 \lesssim \xi_{\epsilon_1}^2 \lesssim \tau_1^2 \xi_\ell^2 \lesssim \tau_3^2 \mu_\ell^2,$$

that is, $\alpha e_{\ell+\epsilon_1}^2 + \beta \mu_{\ell+\epsilon_1}^2 \leq C_9 \tau_3^2 \mu_\ell^2$. The combination of (4.17)-(4.18), for small initial mesh-size $h_8 = \min\{h_7, h_4, (\alpha/\beta C_1 C_4)^{1/2}, (\gamma_0^2/4C_8)^{1/2}\}$ and some simplifications show

$$\beta \mu^2(\mathcal{M}^*) \geq \beta \|h_\ell(f_\ell - f_{\ell+\epsilon_1})\|^2 \geq \left[\frac{\beta \gamma_0^2}{4(2 + C_4 C_1 h_\ell^2)} - 2C_9 \tau_3^2 - \beta \theta_B \right] \mu_\ell^2 + \beta \theta_B \mu_\ell^2.$$

The selections $\tau_3^2 := \frac{\beta \gamma_0^2}{16(2 + C_4 C_1 h_\ell^2) C_9}$ and $\theta_B = \frac{\gamma_0^2}{8(2 + C_4 C_1 h_\ell^2)}$ lead to (4.16). Since \mathcal{M}_ℓ is chosen to be the minimal cardinality set satisfying (4.16), (4.11) and (4.10) yield

$$(4.19) \quad |\mathcal{M}_\ell| \leq |\mathcal{M}^*| \leq |\mathcal{T}_{\ell+\epsilon_1}| - |\mathcal{T}_\ell| \leq |\mathcal{T}_{\epsilon_1}| - |\mathcal{T}_0| \lesssim \epsilon_1^{-\frac{1}{s}} \lesssim \xi_\ell^{-\frac{1}{s}}.$$

Now a combination of both the cases, that is, (4.15) and (4.19) with (4.9) leads to

$$|\mathcal{T}_j| - |\mathcal{T}_0| \lesssim \sum_{\ell=0}^{j-1} |\mathcal{M}_\ell| \lesssim \sum_{\ell=0}^{j-1} \xi_\ell^{-\frac{1}{s}}.$$

A use of the contraction property in Theorem 4.1 shows

$$\begin{aligned} |\mathcal{T}_j| - |\mathcal{T}_0| &\leq (\rho^{j(\frac{1}{2s})} + \rho^{(j-1)(\frac{1}{2s})} + \dots + \rho^{\frac{1}{2s}}) \xi_j^{-\frac{1}{s}} \\ (4.20) \quad &\leq \xi_j^{-\frac{1}{s}} \sum_{j=1}^j \rho^{\frac{j}{2s}} \lesssim \xi_j^{-\frac{1}{s}} \left[\frac{1}{1 - \rho^{\frac{1}{2s}}} \right] \lesssim \xi_j^{-\frac{1}{s}}, \end{aligned}$$

and this concludes the proof. \square

5. Numerical Experiments. This section shows the performance of the adaptive algorithm (AMFEM) on some benchmark problems.

5.1. For the problem (1.1), set $\Omega = (0, 1) \times (0, 1)$ and the coefficients $\mathbf{A} = I$, $\mathbf{b} = (1, 1)$, $\gamma = 2$. Choose the right-hand side f and $u|_{\partial\Omega}$ such that $u = \exp(-100\|x - x_0\|^2)$. The initial uniform criss-cross triangulation \mathcal{T}_0 has mesh-size $h = 0.25$ as shown in Fig. 5.1(a). The adaptive algorithm is performed with parameters $\theta_A = 0.5$, $\theta_B = 0.5$ and $\kappa = 0.8$. Table 5.1 displays the experimental results for errors $\|u - u_\ell\|$ and $\|\mathbf{p} - \mathbf{p}_\ell\|$,

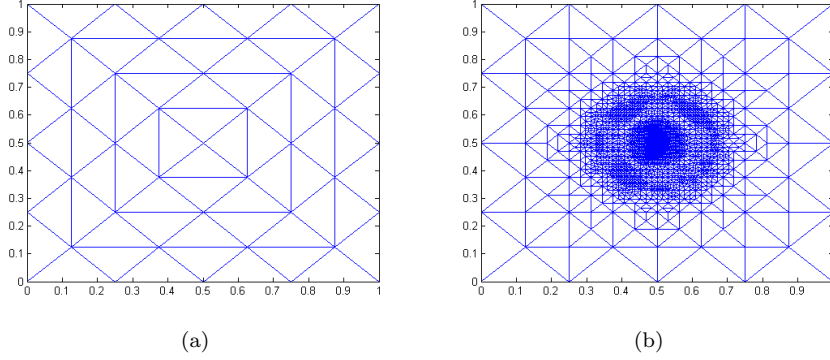


Fig. 5.1: (a) The initial triangulation (b) The adaptive refined mesh at the level 10.

Ndof	$\ u - u_\ell\ $	$\ p - p_\ell\ $	η_ℓ	μ_ℓ	Marking Case
168	0.0571	1.3649	1.5499	4.8186	B
203	0.0435	0.9670	2.7266	4.1602	B
233	0.0336	0.8209	2.9629	3.3392	B
368	0.0254	0.6160	2.2798	2.3559	B
523	0.0197	0.4471	1.5828	1.6907	B
948	0.0152	0.3287	1.1607	1.1389	B
1783	0.0103	0.2451	0.8943	0.8198	B
3373	0.0074	0.1647	0.6013	0.5797	B
6138	0.0058	0.1334	0.4908	0.4182	A
6918	0.0048	0.1027	0.3692	0.3997	B
12053	0.0041	0.0866	0.3120	0.2941	B
21298	0.0033	0.0698	0.2527	0.2177	A
23198	0.0026	0.0547	0.1991	0.2137	B
40658	0.0022	0.0461	0.1685	0.1599	B

Table 5.1: Numerical results of AMFEM for Example 5.1

the edge and volume estimators η_ℓ and μ_ℓ , from (2.4) and (2.5) for several consecutive levels ℓ of AMFEM with the number of degrees of freedom and the marking **Case (A) or (B)**. For this example, the load function f exhibits a relatively large variation in the domain and hence, within the elements. It is observed that the volume estimator μ_ℓ is more than the edge-estimator η_ℓ in several levels due to large data oscillations, which results in the use of the marking **Case (B)** for most of the refinement levels. The reduction of μ_ℓ influences the reduction in the edge-estimator and both lead to the optimal convergence rate $\text{Ndof}^{-1/2}$ for errors, where Ndof denotes the number of degrees of freedom. Fig. 5.1(b) shows the adaptive refined mesh by the AMFEM

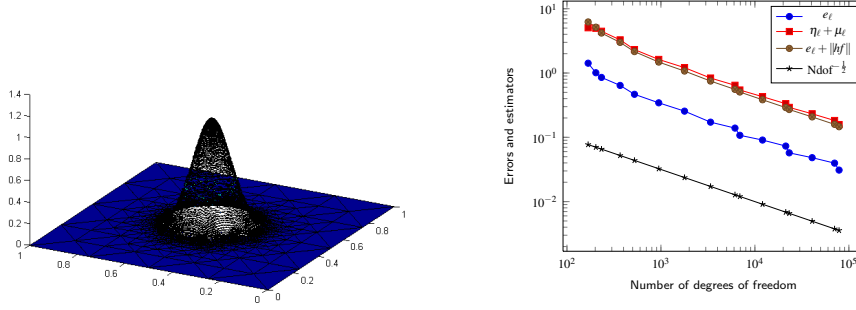


Fig. 5.2: Discrete solution and convergence plot

Ndof	$\ u - u_\ell\ $	$\ p - p_\ell\ $	η_ℓ	μ_ℓ	Marking Case
46	0.0880	0.4021	2.0875	2.5734	A
56	0.0899	0.3979	3.4950	1.9191	B
98	0.0771	0.3557	2.3290	1.8010	B
161	0.0633	0.3085	1.3106	1.4866	A
207	0.0556	0.2643	1.1180	1.0874	B
383	0.0429	0.2401	0.6512	0.8751	A
469	0.0371	0.2042	0.5813	0.6579	A
628	0.0308	0.1695	0.4662	0.4896	A
901	0.0269	0.1400	0.3579	0.3431	B
1436	0.0207	0.1225	0.1971	0.2704	A
1821	0.0183	0.1023	0.1762	0.1949	A
2618	0.0150	0.0844	0.1433	0.1340	B
3957	0.0123	0.0746	0.0795	0.1086	A
5162	0.0111	0.0603	0.0669	0.0698	A
7482	0.0092	0.0486	0.0528	0.0453	B
12145	0.0074	0.0434	0.0282	0.0384	A
15726	0.0065	0.0351	0.0231	0.0246	A
22520	0.0054	0.0280	0.0169	0.0158	B

Table 5.2: Numerical results of AMFEM for Example 5.2

algorithm at the level 10, where the number of degrees of freedom is 6918. Fig. 5.2 depicts the approximate solution u_h and summarises the convergence rates for variables \mathbf{p} , u , and sum of the estimators with respect to the number of degrees of freedom.

5.2. For the Crack problem [2], consider the PDE (1.1) with coefficients $\mathbf{A} = I$, $\mathbf{b} = (x - 1, y + 1)$ and $\gamma = 4$ on $\Omega = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1 \setminus [0, 1] \times \{0\}\}$ with Dirichlet boundary condition and exact solution $u(r, \theta) = r^{1/2} \sin \theta / 2 - r^2 / 2 \sin^2(\theta)$. The initial uniform triangulation \mathcal{T}_0 has mesh-size $h = 0.25$. The adaptive algorithm is performed with parameters $\theta_A = 0.3, \theta_B = 0.3$ and $\kappa = 1$.

Table 5.2 displays the experimental results for the errors $\|u - u_\ell\|$ and $\|\mathbf{p} - \mathbf{p}_\ell\|$, the estimators η_ℓ and μ_ℓ for several consecutive levels ℓ of AMFEM with the number of degrees of freedom and the marking **Case (A) or (B)** for this problem. The right-hand function f of (1.1) in this example is not smooth and the solution has singularity at the origin, and hence the adaptive algorithm utilizes both the marking cases to achieve the optimal convergence rate. Figure 5.3(a) shows optimal convergence

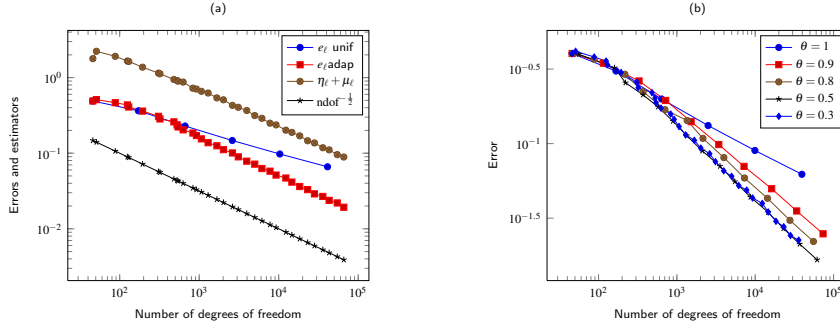


Fig. 5.3: (a) Convergence plots for $\theta_A, \theta_B = 0.3$ (b) Error convergence for $\theta = \theta_A = \theta_B$ equal to 0.3, 0.5, 0.8, 0.9, 1

rate $\text{Ndof}^{-1/2}$ for adaptive algorithm while uniform refinement yields a suboptimal convergence rate $\text{Ndof}^{-1/4}$ for the above mentioned parameters. Figure 5.3(b) shows the convergence rate of the error for different values of θ_A and θ_B . This supports the theoretical prediction of the choice of marking parameters θ_A and θ_B in the proof of Theorem 4.5, that is, the quasi-optimality can be achieved only for small values of θ_A and θ_B .

5.3. Consider the PDE (1.1) with coefficients $\mathbf{A} = I$, $\mathbf{b} = (0, 0)$, $\gamma = -8.9$, Dirichlet boundary condition on the L-shaped domain $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1] \times [-1, 0]$ and the exact solution given in polar coordinates as $u(r, \theta) = r^{2/3} \sin(2\theta/3)$. The numerical experiment is performed with initial mesh-size $h = 0.25$ and uni-

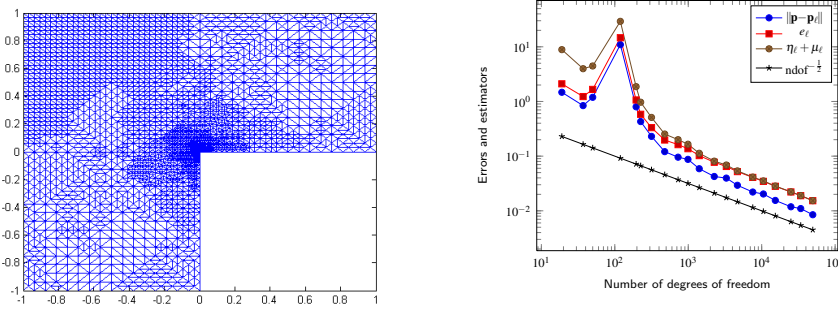


Fig. 5.4: Adaptive mesh-refinement and the convergence plot

form triangulation. The adaptive algorithm is performed with parameter choice $\theta_A = 0.5, \theta_B = 0.5$ and $\kappa = 2$. Since the forcing function f is smooth, the oscillation term has higher order convergence than the remaining terms in the estimator. In this example, the marking is based on the edge estimator to obtain the optimal convergence. The refinement due to the data oscillations and the volume estimators can be avoided since they play a minor role in the convergence. Figure 5.4 displays the adaptive mesh-refinement and the optimal convergence rate $\text{Ndof}^{-1/2}$ of the errors and the estimators for a sufficiently small mesh-size h .

5.4. Observations. This subsection deals with a few observations.

- If the given function f has large variation in the domain and within the elements, then adaptive algorithm chooses **Case (B)** marking criteria and achieves the convergence (see 5.1).
- When f is not smooth and the solution also has singularity, then the adaptive algorithm utilizes both the marking cases to achieve the optimal convergence rate (see 5.2).
- For the smooth function f , the oscillation has higher order convergence than the remaining estimator terms. In such cases, the data oscillations and the volume estimators have a very minor role in the convergence. Thus, the marking would be based on the edge-based error estimator to capture singularity of the solution (see 5.3).

6. Conclusions. In this work, the convergence and the quasi-optimality of adaptive mixed finite element method is analysed for the non-symmetric and indefinite second order elliptic equations using the lowest order Raviart-Thomas elements. The adaptive algorithm in Subsection 2.1 is designed as a combination of the edge and the volume error estimators. The numerical experiments confirm the efficiency of this algorithm and support the theoretical findings.

Acknowledgements. The first author acknowledges the financial support of Council of Scientific and Industrial Research (CSIR), Government of India. The authors sincerely thank Professor Carsten Carstensen, Humboldt University, Berlin for his constructive comments and suggestions.

REFERENCES

- [1] M. AINSWORTH AND J. T. ODEN, *A posteriori error estimation in finite element analysis*, Wiley, New York, 2000.
- [2] C. BAHRIAWATI AND C. CARSTENSEN, *Three matlab implementation of the lowest order Raviart-Thomas MFEM with a posteriori error control*, Comput. Methods Appl. Math., 5 (2005), pp. 333-1361.
- [3] S. C. BRENNER AND L. R. SCOTT, *The mathematical theory of finite element methods*, third ed., Texts in Applied Mathematics, 15, Springer, New York, 2008.
- [4] R. BECKER AND S. MAO, *An optimally convergent adaptive mixed finite element method*, Numer. Math., 111 (2008), pp. 35-54.
- [5] R. BECKER AND S. MAO, *A convergent nonconforming adaptive finite element method with quasi-optimal complexity*, SIAM J. Numer. Anal., 47 (2010), pp. 4639-4659.
- [6] P. BINEV, W. DAHMEN, AND R. DEVORE, *Adaptive finite element methods with convergence rates*, Numer. Math., 97 (2004), pp. 219-268.
- [7] F. BREZZI AND M. FORTIN, *Mixed and hybrid finite element methods*, Springer Verlag, 1991.
- [8] C. CARSTENSEN, *A posteriori error estimate for the mixed finite element method*, Math. Comp., 66 (1997), pp. 465-476.
- [9] C. CARSTENSEN, *A unifying theory of a posteriori finite element error control*, Numer. Math., 100 (2005), pp. 617-637.
- [10] C. CARSTENSEN, *Convergence of adaptive finite element methods in computational mechanics*, Appl. Numer. Math., 59 (2009), pp. 2119-2130.
- [11] C. CARSTENSEN AND J. HU, *A unifying theory of a posteriori error control for nonconforming finite element methods*, Numer. Math., 107 (2007), pp. 473-502.
- [12] C. CARSTENSEN, ASHA. K. DOND, N. NATARAJ, AND AMIYA. K. PANI, *Error analysis of non-conforming and mixed FEMs for second-order linear non-selfadjoint and indefinite elliptic problems*, <http://arxiv.org/abs/1401.4810>.
- [13] C. CARSTENSEN AND R. HELLA, *An optimal adaptive mixed finite element method*, Math. Comp., 80 (2011), pp. 649-667.
- [14] C. CARSTENSEN AND R.H.W. HOPPE, *Convergence analysis of adaptive nonconforming finite element methods*, Numer. Math., 103 (2006), pp. 251-266.

- [15] C. CARSTENSEN AND R. H. W. HOPPE, *Error reduction and convergence for an adaptive mixed finite element method*, Math. Comp., 75 (2006), pp. 1033-1042.
- [16] C. CARSTENSEN, M. FEISCHL, M. PAGE, AND D. PRAETORIUS, *Axioms of Adaptivity*, ASC Report 38/2013, Institute for Analysis and Scientific Computing, Vienna University of Technology.
- [17] J. CASCON, C. KREUZER, R. H. NOCHETTO, AND K. G. SIEBERT, *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal., 46 (2008), pp. 2524-2550.
- [18] J. CASCON, R. H. NOCHETTO, AND K. G. SIEBERT, *Design and convergence of AFEM in $H(\text{div})$* , Math. Models Methods Appl. Sci., 17 (2007), pp. 1849-1881.
- [19] J. CHEN AND L. LI, *Convergence and domain decomposition algorithm for nonconforming and mixed methods for nonselfadjoint and indefinite problems*, Comput. Methods Appl. Engrg., 173 (1999), pp. 1-20.
- [20] L. CHEN, M. HOLST, AND J. XU, *Convergence and optimality of adaptive mixed finite element methods*, Math. Comp., 78 (2009), pp. 35-53.
- [21] H. CHEN, X. XU, AND R.H.W. HOPPE, *Convergence and Quasi-optimality of adaptive nonconforming finite element methods for some nonsymmetric and indefinite problems*, Numer. Math., 116 (2010), pp. 383-419.
- [22] A. DEMLOW AND R. STEVENSON, *Convergence and quasi-optimality of an adaptive finite element method for controlling L_2 errors*, Numer. Math., 117 (2011), pp. 185-218.
- [23] J. JR. DOUGLAS AND J.E. ROBERT, *Global estimates for mixed methods for second order elliptic equations* Math. Comput. 44(1985), pp. 39-51.
- [24] M. FEISCHL, T. FÜHRER, AND D. PRAETORIUS, *Adaptive FEM with optimal convergence rates for a certain class of non-symmetric and possibly non-linear problems*, ASC Report 43/2012, Institute for Analysis and Scientific Computing, Vienna University of Technology.
- [25] K. MEKCHAY AND R.H. NOCHETTO, *Convergence of adaptive finite element methods for general second order elliptic PDE*, SIAM J. Numer. Anal., 43 (2005), pp. 1803-1827.
- [26] L. D. MARINI, *An inexpensive method for the evaluation of the solution of the lowest order Raviart-Thomas mixed method*, SIAM J. Numer. Anal., 22 (1985), pp. 493-496.
- [27] W. DÖRFLER, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. 33 (1996), pp. 1106-1124.
- [28] P. MORIN, R.H. NOCHETTO, AND K.G. SIEBERT, *Data oscillation and convergence of adaptive fem*, SIAM J. Numer. Anal., 38 (2000), pp. 466-488.
- [29] P. MORIN, R.H. NOCHETTO, AND K.G. SIEBERT, *Convergence of adaptive finite element methods*, SIAM Rev., 44 (2002), pp. 631-658.
- [30] P. MORIN, K. G. SIEBERT, AND A. VEESER, *A basic convergence result for conforming adaptive finite elements*, Math. Models Methods Appl. Sci., 18 (2008), pp. 707-737.
- [31] A. H. SCHATZ, *An observation concerning Ritz-Galerkin methods with indefinite bilinear forms*, Math. Comput., 28 (1974), pp. 959-962.
- [32] A. H. SCHATZ AND J. WANG, *Some new error estimates for Ritz-Galerkin methods with minimal regularity assumptions*, Math. Comp., 65 (1996), pp. 19-27.
- [33] R. STEVENSON, *Optimality of a standard adaptive finite element method*, Found. Comput. Math., 7 (2007), pp.245-269.
- [34] R. VERFÜRTH, *A posteriori error estimation and adaptive mesh-refinement techniques*, J. Comput. Appl. Math., 50 (1994), pp. 67-83.

Appendix I

List of the constants

Constant	Dependency on	Appears first in	Value
I. Natural Constants			
C_{rel}	Coefficients of (1.1) and interpolation constants	Theorem 3.2 (<i>A posteriori estimates</i>)	Positive
ϵ	Coefficients of (1.1) and interpolation constants		Positive
h_1	ϵ		Positive
C_{eff}	Coefficients of (1.1), inverse inequality and finite overlap	Lemma 3.3 (<i>efficiency</i>)	Positive
C_1	Coefficients of (1.1)	Lemma 3.4 (<i>Volume estimator reduction</i>)	Positive
C_2	Coefficients of (1.1), inverse inequality and finite overlap	Lemma 3.6 (<i>edge estimator reduction</i>)	Positive
C_3	Coefficients of (1.1)	Lemma 3.7 (<i>Quasi-orthogonality</i>)	Positive

Continued on next page

Table 6.1 – continued from previous page

Constant	Dependency on	Appears first in	Value
C_4	Coefficients of (1.1), interpolation constants	Lemma 3.9 (<i>Quasi-reliability</i>)	Positive
γ_0	$\begin{cases} 0 < \gamma_0 < 1 & \text{if } T \in \mathcal{T}_H, \text{ refined} \\ 1 & \text{otherwise} \end{cases}$	Corollary 3.5	Positive
II. Contraction property			
θ_B	$0 < \theta_B < 1$, to be chosen	Lemma 3.4 (<i>volume-estimator reduction</i>)	(0,1)
δ_1	$< \frac{\theta_B}{4-\theta_B}$		(0,1)
ρ_B	$(1 + \delta_1)(1 - \frac{\theta_B}{4})$		(0,1)
A_1	$C_1(1 + \frac{1}{\delta_1})$		(0,1)
θ_A	$0 < \theta_A < 1$, to be chosen	Lemma 3.6 (<i>edge-estimator reduction</i>)	(0,1)
δ_2	$\frac{\theta_A}{4-2\theta_A}$		(0,1)
A_2	$C_2(1 + \frac{1}{\delta_2})$		(0,1)
ρ_A	$(1 + \frac{1}{\delta_2})(1 - \frac{\theta_A}{2})$		(0,1)
C_3	Coefficient of (1.1)	Lemma 3.6 (<i>Quasi-orthogonality</i>)	Positive
α_2	$0.5 + 7C_3^2 A_1 H^2$		(0,1)
α_3	$0.5 - 7C_3^2 A_1 H^2$		(0,1)
δ_3	< 1		(0,1)
α_1	$\delta_3 + 4C_3^2 \epsilon^2$	Remark 3.2	(0,1)
α_4	$\frac{1}{2} \min\{1, \frac{(1-\rho_A)\alpha_3}{(A_1+A_2)C_{rel}}\}$		(0,1)
δ_3	$< \alpha_4$		Positive
ϵ	$(\alpha_4 - \delta_3)/(4C_3^2)$		Positive
h_2	ϵ and $< \frac{1}{\sqrt{14C_3^2 A_1}}$		(0,1)
α_1	$< \alpha_4$		(0,1)
C_5	$(A_1 + A_2)/\alpha_3$	Theorem 4.1 (<i>Contraction property</i>)	Positive
α	$C_5(1 - \alpha_1)$		Positive
h_3	$\min\{h_1, h_2, \frac{1}{\sqrt{\beta}}\}$		Positive
E	$2(C_5 A_3 + C_5 \alpha_4 C_{rel})$		Positive
κ	$< \kappa_0 := \frac{(1-\rho_A - C_5 \alpha_4 C_{rel})}{\beta}$		Positive
D	$(1.5 + C_5 \alpha_4 C_{rel})(\kappa)$		Positive
δ_1	$\min\{\frac{E}{2\beta}, \frac{\theta_B}{4-\theta_B}\}$		(0,1)
β	$2 \max\{C_5 A_3 + C_5 \alpha_4 C_{rel}, \frac{C_5 A_3 + C_5 \alpha_4 C_{rel} + D}{1-\rho_B}\}$		Positive
ρ_1	$\max\left\{\frac{C_5(1-\alpha_4)}{\alpha}, \rho_A + C_5 \alpha_4 C_{rel} + E\kappa, \frac{((1+\delta_1)\beta - E/2)}{\beta}\right\}$		(0,1)
ρ_2	$\max\left\{\frac{C_5(1-\alpha_4)}{\alpha}, 2 + C_5 \alpha_4 C_{rel} - D\kappa, \frac{\rho_B \beta + C_5 A_3 + C_5 \alpha_4 C_{rel} + D}{\beta}\right\}$		(0,1)
ρ	$\max\{\rho_1, \rho_2\}$		(0,1)
III. Quasi-Optimality			
h_4	$\frac{1}{2(7+C_3 A_1 + \beta A_1)/(C_{eff}^{-1} + \alpha)}$	Lemma 4.3	Positive
θ_A	$\min\{1, \frac{C_{eff}}{9C_4}\}$	Theorem 4.5 (<i>Quasi-optimality</i>)	(0,1)
θ_B	$(\gamma_0^2)/(8(2 + C_4 C_1 h^2))$		(0,1)
κ	$\min\{\kappa_0, \frac{C_{eff}/8 - C_4 \theta_A}{4C_4 + A_3/2}\}$		Positive
ϵ	$\min\{\epsilon, 1, \frac{1}{4C_4}\}$		Positive
h_5, h_7	ϵ		Positive
τ_2^2	$\frac{(C_{eff}/8) - \kappa(4C_4 + A_3/2) - C_5 \theta_A}{(5/4)C_7}$		Positive
h_6	ϵ and $\min\{h_2, h_4, \frac{1}{\sqrt{4C_4 A_1}}\}$		Positive
τ_3^2	$\frac{\beta \gamma_0^2}{8(2+C_4 C_1 h^2)C_8}, C_8 = C_4 C_1 (C_{rel}+1)(1 + \frac{1}{\kappa})$		Positive
h_8	ϵ and $\min\{h_2, h_4, \sqrt{\frac{\alpha}{\beta C_1 C_4}}, \sqrt{\frac{\gamma_0^2}{4C_8}}\}$		Positive
h_0	$\min\{h_7, h_8\}$		Positive
ϵ_1	$\tau_1 \xi_\ell$, where $\tau_1 := \min\{\tau_2, \tau_3\}$		Positive